CHAPTER FIVE
Network Hydraulics

The fundamental relationships of conservation of mass and energy mathematically describe the flow and pressure distribution within a pipe network under steady state conditions. To begin, parallel and series pipe systems are considered in Section 5.1. The basic concepts applied in these simple systems to determine the pipe flow rates and nodal pressure heads are then extended to full networks. Four mathematical formulations are discussed (Sections 5.2 and 5.3). Quasi-dynamic flow for tank modeling under time varying conditions is then presented. Finally, flow conditions that change in the short term are analyzed using conservation of mass with conservation of momentum to include the impact of dynamic changes.

Notation for this section can become confusing since a number of subscripts and counters are used. To summarize, the subscripts i and l identify nodes and pipes, respectively. The pipe flows are typically summed over the sets of pipes providing flow to or carrying flow from a node. To avoid confusion with pipe lengths, L, the set of incoming and outgoing pipes are defined as J_{in} and J_{out}, respectively. Depending upon the conservation of energy formulation, head losses in pipes are summed over the set of pipes in a loop, *lloop*, or path, *lpath*. The notation with the letter n followed by an identifier is used for the number of components such as *nnode*, *npipe*, *nloop*, and *nploop* for the number of network nodes, pipes, closed loops, and pseudo-loops, respectively. The subscript j is used as an identifier in summations and defined for the specific equation including an identifier for a node at one end of a pipe. To compute the nodal heads and pipe flows a set of nonlinear equations is solved by an iterative process. The counter m is used to define the iteration number and added to the unknown variable as a superscript in parentheses, i.e., $Q^{(m)}$. Lastly, variables are shown in italics and matrices and vectors are denoted by bold characters.

5.1 SIMPLE PIPING SYSTEMS

Simple piping systems provide insight into the understanding of pipe networks. Variation of total head through a network is seen in a set of pipes in series. Analysis of pipes in parallel is the first application of the conservation of mass at a junction and conservation of energy around a loop. In addition to the physical understanding that they offer, simplification of these systems to equivalent pipes reduces computations during analysis.

5.1.1 Pipes in Series

As shown in Figure 5-1, a series of pipes may have different pipe diameters and/or roughness parameters. The total head loss is equal to the sum of the head loss in each pipe segment or:

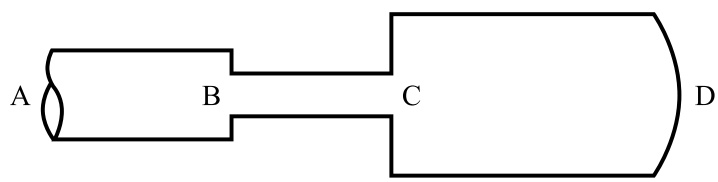
$$h_L = \sum_{l=1}^{lpath} h_{L,l} = \sum_{l=1}^{lpath} K_l\, Q_l^n \tag{5-1}$$

where *lpath* is the number of pipes in series, K_l is the coefficient for pipe *l* containing information about the diameter, length, and pipe roughness, *n* is the exponent from the head loss equation, and Q_l is the flow rate in pipe *l*.

If no withdrawals occur along the pipe, each pipe carries the same flow rate but the rate of head loss in each pipe may be different. If we use the same head loss equation for all pipes (i.e., the same *n*), we can take *Q* out of the summation or:

$$h_L = \sum_{l=1}^{lpath} K_l\, Q_l^n = \sum_{l=1}^{lpath} K_l\, Q^n = Q^n \sum_{l=1}^{lpath} K_l = K_{eq}^s\, Q^n \tag{5-2}$$

where K_{eq}^s is the equivalent *K* coefficient for series of pipes. If flow is turbulent, the K_l's are constant and a single equivalent K_{eq}^s can be computed for all turbulent flows.



D_1 = 30 cm D_2 = 20 cm D_3 = 40 cm
L_1 = 2000 m L_2 = 1000 m L_3 = 2000 m
f_1 = 0.022 f_2 = 0.025 f_3 = 0.021
z_A = 20 m, z_B = 25 m, z_C = 32.5 m, z_D = 37.5 m

Figure 5-1: Pipes in series with data for Example 5.1.

Example 5.1

Problem: For a flow rate of 0.04 m^3/s, determine the pressure and total heads at points A, B, C, and D for series pipes shown in Figure 5-1. Assume fully turbulent flow for all cases and the pressure head at point A is 40 m.

Solution: The pressure and total heads are computed using the energy equation along the path beginning at point A. Given the pressure head and elevation, the total head at point A is:

$$H_A = \frac{p_A}{\gamma} + z_A = 40 + 20 = 60\ m$$

Note the velocity and velocity head are:

$$V_1 = \frac{Q}{A} = \frac{0.04}{\frac{\pi(0.3)^2}{4}} = 0.57\ m/s$$

and

$$\frac{V_1^2}{2g} = \frac{0.57^2}{2(9.81)} = 0.017\ m$$

The velocity head is four orders of magnitude less than the static head so it can be neglected. Neglecting velocity head is a common assumption in pipe network analysis.

All energy loss in the system is due to friction. So following the path of flow the total heads at A, B, C, and D are:

$$H_B = H_A - h_f^{A-B} = 60 - f_1 \frac{L_1}{D_1}\frac{V_1^2}{2g} = 60 - 0.022\left(\frac{2000}{0.3}\right)\frac{0.57^2}{2(9.81)}$$
$$= 60 - 2.4 \Rightarrow H_B = 57.6\ m$$

$$H_C = H_B - h_f^{B-C} = 57.6 - f_2 \frac{L_2}{D_2}\frac{V_2^2}{2g} = 57.6 - 0.025\left(\frac{1000}{0.2}\right)\frac{\left(\frac{0.04}{\pi(0.2)^2/4}\right)^2}{2(9.81)} =$$
$$= 57.6 - 10.3 \Rightarrow H_C = 47.3\ m$$

$$H_D = H_C - h_f^{C-D} = 47.3 - f_3 \frac{L_3}{D_3}\frac{V_2^2}{2g} = 47.3 - 0.021\left(\frac{2000}{0.4}\right)\frac{\left(\frac{0.04}{\pi(0.4)^2/4}\right)^2}{2(9.81)} =$$
$$= 47.3 - 0.5 \Rightarrow H_D = 46.8\ m$$

The pressure heads are:
at point B

$$H_B = \frac{p_B}{\gamma} + z_B = 57.6 = \frac{p_B}{\gamma} + 25 \Rightarrow \frac{p_B}{\gamma} = 32.6\ m$$

at point C

$$H_C = \frac{p_C}{\gamma} + z_C = 47.3 = \frac{p_C}{\gamma} + 32.5 \Rightarrow \frac{p_C}{\gamma} = 14.8\ m$$

and at point D

$$H_D = \frac{p_D}{\gamma} + z_D = 46.8 = \frac{p_D}{\gamma} + 37.5 \Rightarrow \frac{p_D}{\gamma} = 9.3\ m$$

Example 5.2

Problem: For the series pipe system in Example 5.1, find the equivalent roughness coefficient and the total head at point D for a flow rate of 0.03 m^3/s.

Solution: By Eq. 5-2, the equivalent pipe loss coefficient is equal to the sum of the pipe coefficients or:

$$K_{eq}^s = \sum_{l=1}^{3} K_l = \sum_{l=1}^{3} \frac{8 f_l L_l}{g \pi^2 D_l^5}$$

For this problem:

$$K_{eq}^s = K_1 + K_2 + K_3 = \frac{8\ f_1\ L_1}{g\ \pi^2\ D_1^5} + \frac{8\ f_2\ L_2}{g\ \pi^2\ D_2^5} + \frac{8\ f_3\ L_3}{g\ \pi^2\ D_3^5}$$
$$= \frac{8(0.022)(2000)}{9.81\pi^2\ (0.3)^5} + \frac{8(0.025)(1000)}{9.81\pi^2\ (0.2)^5} + \frac{8(0.021)(2000)}{9.81\pi^2\ (0.4)^5}$$
$$= 1496 + 6455 + 339 \Rightarrow K_{eq}^s = 8290$$

Note that pipe 2 has the largest loss coefficient since it has the smallest diameter and highest flow velocity. As seen in Example 5.1, although it has the shortest length, most of the head loss occurs in this section. The head loss between nodes A and D for $Q = 0.03\ m^3/s$ is then:

$$h_f^{A-D} = K_{eq}^s\ Q^2 = K_1\ Q^2 + K_2\ Q^2 + K_3\ Q^2 = 8290\ (0.03)^2 \Rightarrow h_f^{A-D} = 7.5\ m$$

We can also confirm the result in Example 5.1 by substituting $Q = 0.04\ m^3/s$ in which case:

$$h_f^{A-D} = K_{eq}^s\ Q^2 = 8290\ (0.04)^2 = 13.26\ m$$

and

$$H_D = H_A - h_f^{A-D} = 60 - 13.26 \Rightarrow H_D = 46.74\ m$$

that would be equivalent to the earlier result if Example 5.1 was carried to 2 decimal places.

5.1.2 Pipes in Parallel

When one or more pipes connect the same locations (junctions), the hydraulics are much more interesting. The relationships in these small networks lead to the fundamental relationships for full network modeling. Locations A and B in Figure 5-2 are described as nodes or junctions of several pipes. As in Example 2.1, conservation of mass must be preserved at these locations. That is, in steady state the known incoming flow at node A must balance with the outgoing flows in pipes 1, 2, and 3. Similarly, the incoming flows to node B in the incoming pipes 1, 2, and 3 must equal the known withdrawal at node B.

$$q_A = q_B = Q_1 + Q_2 + Q_3 \tag{5-3}$$

where Q_l and q_j define the flow rate in pipe l and the nodal withdrawal/supply at node j, respectively. The mass balance for node B provides the same information as the above and is redundant.

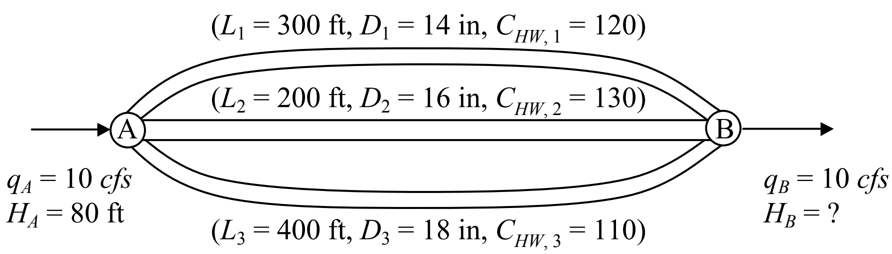


Figure 5-2: Pipes in parallel with data for Example 5.3.

The second relationship that must hold is that the head loss in pipes 1, 2, and 3 must be the same. Since all begin at a single node (A) and all end at a single node (B) and the difference in head between those two nodes is unique, regardless of the pipe characteristics the head loss in the pipes is the same or:

$$H_A - H_B = h_{L,1} = h_{L,2} = h_{L,3} = h_L \tag{5-4}$$

where H_A and H_B are the total heads at nodes A and B, respectively, $h_{L,l}$ is the head loss in pipe l, and h_L is the single value of head loss between nodes A and

B. Eq. 5-4 is a statement of conservation of energy for a pipe and is used in several formulations for solving for flows and heads in a general network.

In other network solution methods, we write conservation of energy for closed loops. A closed loop is a path of pipes that begins and ends at the same node. Pipes 1 and 2 form a closed loop beginning and ending at node A. Starting at node A around the path, energy, $h_{L,1}$, is lost as water flows from A to B. As we follow the path back to node A to close the loop, we gain energy, $h_{L,2}$, since we are moving in the direction opposite to the flow. We can write the path equation around the loop and manipulate it to show:

$$H_A - h_{L,1} + h_{L,2} = H_A \Rightarrow -h_{L,1} + h_{L,2} = 0 \Rightarrow h_{L,1} = h_{L,2} \qquad (5\text{-}5)$$

Now using Eq. 5-3 and either Eqs. 5-4 or 5-5, we can determine the head loss and flow for each pipe and an equivalent pipe coefficient, K_{eq}^{p}. In any pipe network system the nodal inflows and outflows (q_A and q_B) and at least one nodal head's total energy (H_A in this case) must be known to provide a datum for the pressure head. For steady flow conditions in the network in Figure 5-2, we have a total of seven unknowns, node B's total energy (H_B), three pipe flows (Q_1, Q_2, and Q_3) and three head losses ($h_{L,1}$, $h_{L,2}$, and $h_{L,3}$).

Eq. 5-4 provides two independent equations relating the head losses ($h_{L,1} = h_{L,2}$, and $h_{L,1}, = h_{L,3}$). The third equation is that the head loss in any pipe equals the difference in head between nodes A and B (the first part of Eq. 5-4). Conservation of mass at node A (Eq. 5-3) is the fourth relationship. The final three equations are the head loss versus discharge equations:

$$h_{L,l} = K_l\, Q_l^n \text{ or } Q_l = \left(\frac{h_{L,l}}{K_l}\right)^{1/n} \qquad (5\text{-}6)$$

We can substitute Eq. 5-6 in the mass balance equations (Eq. 5-3) with h_L equal to each pipe's head loss or:

$$\left(\frac{h_L}{K_1}\right)^{1/n} + \left(\frac{h_L}{K_2}\right)^{1/n} + \left(\frac{h_L}{K_3}\right)^{1/n} = q_A \qquad (5\text{-}7)$$

In this equation, all terms except for h_L are known. After solving for h_L, the unknown pipe flows can be computed by Eq. 5-6 and H_B can be determined in Eq. 5-4.

Like pipes in series, an equivalent pipe coefficient can be computed for parallel pipes. In Eq. 5-7, h_L can be pulled from each term on the left hand side or for a general discharge and three parallel pipes:

$$\left[\left(\frac{1}{K_1}\right)^{1/n}+\left(\frac{1}{K_2}\right)^{1/n}+\left(\frac{1}{K_3}\right)^{1/n}\right]h_L^{1/n}=q \qquad (5\text{-}8)$$

The equivalent coefficient is then:

$$\left(\frac{1}{K_1}\right)^{1/n}+\left(\frac{1}{K_2}\right)^{1/n}+\left(\frac{1}{K_3}\right)^{1/n}=\left(\frac{1}{K_{eq}^{p}}\right)^{1/n}=\sum_{l=1}^{lp}\left(\frac{1}{K_l}\right)^{1/n} \qquad (5\text{-}9)$$

where K_{eq}^{p} is the equivalent pipe coefficient for parallel pipes. As shown in the last term, Eq. 5-9 can be generalized for *lp* parallel pipes.

The head loss between the two end nodes is:

$$h_L = K_{eq}^{p}\,(Q_{total})^{n} \qquad (5\text{-}10)$$

Example 5.3

Problem: Given the data for the three parallel pipes in Figure 5-2, compute (1) the equivalent parallel pipe coefficient, (2) the head loss between nodes A and B, (3) the flow rates in each pipe, and (4) the total head at node B.

Solution: (1) The equivalent parallel pipe coefficient allows us to determine the head loss that can then be used to disaggregate the flow between pipes. The loss coefficient for the Hazen-Williams equation for pipe 1 with English units is:

$$K_1=\frac{4.73\,L_1}{C_1^{1.85}\,D_1^{4.87}}=\frac{4.73\,(300)}{120^{1.85}\,(14/12)^{4.87}}=0.0954$$

Similarly, K_2 and K_3 equal 0.0286 and 0.0439, respectively.

The equivalent loss coefficient is:

$$\left(\frac{1}{K_1}\right)^{1/n}+\left(\frac{1}{K_2}\right)^{1/n}+\left(\frac{1}{K_3}\right)^{1/n}=\left(\frac{1}{0.0954}\right)^{0.54}+\left(\frac{1}{0.0286}\right)^{0.54}+\left(\frac{1}{0.0439}\right)^{0.54}=$$

$$=3.557+6.817+5.408=\left(\frac{1}{K_{eq}^{p}}\right)^{0.54}\Rightarrow K_{eq}^{p}=0.00607$$

(2 and 4) The head loss between nodes A and B is then:

$$h_L = K_{eq}^p (Q_{total})^n = 0.00607 (10)^{1.85} = 0.43 \; ft$$

(4) So the head at node B, H_B, is:

$$H_A - H_B = h_L = 80 - H_B = 0.43 \, ft \;\; \Rightarrow \; H_B = 79.57 \, ft$$

(3) The flow in each pipe can be computed from the individual pipe head loss equations since the head loss is known for each pipe ($h_L = 0.43 \, ft$).

$$Q_1 = \left(\frac{1}{K_1} \right)^{1/n} h_L^{1/n} = \left(\frac{1}{0.0954} \right)^{0.54} (0.43)^{0.54} = 2.26 \;\; cfs$$

The flows in pipes 2 and 3 can be computed by the same equation and are 4.32 and 3.43 cfs, respectively. The sum of the three pipe flows equals 10 cfs, which is same as inflow to node A.

5.2 SYSTEM OF EQUATIONS FOR STEADY FLOW

Conservation of mass at a junction node (Eq. 5-3) and conservation of energy (Eq. 5-1) can be extended from parallel pipes to general networks for steady state hydraulic conditions. The resulting set of simultaneous quasi-linear equations can be solved for the pipe flows and nodal heads for steady state and step-wise (quasi) dynamic (known as extended period simulation or EPS) analyses. EPS analysis requires an additional relationship describing changes in tank levels due to inflow/outflows and is discussed in later sections. Only steady state hydraulics is considered in Sections 5.2 and 5.3.

5.2.1 Conservation of Mass

As defined earlier, a junction node is a connection of two or more pipes. Although demands are distributed along pipes, these demands are lumped at junctions and defined as q_{node}. Conservation of mass at a node was presented in Section 2.1.2.1. For a junction node i, conservation of mass can be written as:

$$\sum_{l \in J_{in}} Q_l - \sum_{l \in J_{out}} Q_l = q_i \qquad (5\text{-}11)$$

where q_i is the external demand (withdrawal), $J_{in,i}$ and $J_{out,i}$ are the set of pipes supplying and carrying flow from node i, respectively, and $l \in J_{in}$ denotes that l is in the set of pipes in J_{in}. This equation can be written for every junction node in the system.

5.2.2 Conservation of Energy

The second governing law is that energy must be conserved between any two points. Along the path between nodes A and B that only includes pipes, conservation of energy is written as:

$$H_A - H_B = \sum_{i \in lpath} h_{L,l} = \sum_{i \in lpath} K_l \lfloor Q_l \rfloor^n \qquad (5\text{-}12)$$

where H_A and H_B are the total energy at nodes A and B, $h_{L,l}$, K_l, Q_l are the head loss, loss equation coefficient and flow rate in pipe l and n is the exponent from the head loss equation.

The sign for the flow rate is defined using the $\lfloor Q_l \rfloor$ symbol and this symbol does not have its conventional meaning. This symbol is intended as a short form notation and reminder of how the signs of this relationship should be interpreted. The absolute value of Q is raised to the power of n and the sign of the pipe term is based on the flow direction. If flow is moving from node A toward node B then the sign should be taken positive and a negative sign is used if flow is away from B toward A. Eq. 5-12 can be written for a closed or pseudo-loop or a single pipe. *lpath* defines the set of pipes in the path.

A *closed loop* is one that begins and ends at the same node. Since each location in the network has a unique energy the net energy loss around a closed loop is zero. For a loop beginning and ending at node A:

$$H_A - H_A = 0 = \sum_{l \in lloop} h_{L,l} = 0 \qquad (5\text{-}13)$$

where *lloop* is the set of pipes in the closed loop.

A *pseudo-loop* is a path of pipes between two points of known energy such as two tanks or reservoirs. Eq. 5-12 applies directly to pseudo-loops. Pseudo-loop equations include additional information regarding the flow distribution and are needed for some solution methods. Finally, Eq. 5-12 also applies directly for individual pipes with H_A and H_B being the total heads at the two ends of the single pipe or:

$$H_A - H_B = K_l \lfloor Q_l \rfloor^n \qquad (5\text{-}14)$$

5.2.3 Systems of Equations

The unknowns in a steady state hydraulic analysis are the flows in each pipe, Q, and the total energy head at each junction node, H. In a system with *nnode* nodes and *npipe* pipes, the total number of unknowns is *nnode* + *npipe*. Four equation formulations can be developed to solve for these unknowns. They can be expressed in terms of unknown pipe flows or nodal heads. All sets are nonlinear due to the energy loss relationships and require iterative solutions. The Newton-Raphson method is the most widely used iterative solution procedure in network analysis. Its convergence properties have been studied in detail by Altman and Boulos (1995). At least one point of known energy is required to provide a datum (or root) for the nodal heads. The four solution approaches are summarized below and mathematical details are presented in Section 5.3.

5.2.3.1 Loop Equations

The smallest set of equations is the loop equations that include one equation for each closed loop and pseudo-loop for a total of *nloop* + *nploop* equations where *nloop* and *nploop* are the number of closed and pseudo-loops, respectively. The unknowns in the loop equations are ΔQ*'s* that are defined as the corrections to the flow rate around each loop. Beginning with a flow distribution that satisfies conservation of mass, the corrections maintain those relationships. When zero corrections are needed in all loops, the flow rates in each loop and each pipe has been found. After the flows have been determined, Eq. 5-12 is applied beginning at a location of known total energy (e.g., root) to determine the nodal heads.

The Hardy Cross method is one approach to solve the loop equations. This method first determines corrections for each loop independently then applies the corrections to compute the new pipe flows. With the new flow distribution, another set of corrections is computed. Hardy Cross introduced this method in 1936 and, although amendable to hand calculations, it is not efficient compared to methods that consider the entire system simultaneously. Epp and Fowler (1970) presented a more efficient method that simultaneously solves for all loop corrections using the Newton-Raphson method with the corrections as the unknowns.

5.2.3.2 Node-Loop Equations

Wood and Rayes (1981) compared a number of solution algorithms with their modified linear theory (flow adjustment) method and showed that this approach was efficient and robust. Modified linear theory solves directly for the pipe flow rates rather than the loop equation approach of loop flow corrections.

The *nloop* + *nploop* loop equations (Eq. 5-12) incorporate the concept of energy conservation and *nnode* node equations (Eq. 5-11) introduce conservation of mass. The total number of independent equations is *nnode* + *nloop* + *nploop* that number is equal to the number of unknown pipe flow rates, *npipe*. A Newton's type method is used to solve for the Q's directly rather than the loop flow rate corrections, ΔQ*'s*. As in the loop equations, after the pipe flow rates are found, they are substituted in Eq. 5-12 beginning at a point of known energy to compute the nodal heads.

5.2.3.3 Node Equations

The node equations can be rewritten in terms of the nodal heads by writing Eq. 5-14 for pipe l that connects nodes j and i as:

$$Q_l = \left(\frac{\lfloor H_j - H_i \rfloor}{K_l} \right)^{1/n} \tag{5-15}$$

These terms are substituted for the flow rate in Eq. 5-11 for each pipe and one equation of the form of Eq. 5-15 is written for each node. This substitution combines the conservation of energy and mass relationships resulting in *nnode* equations in terms of the *nnode* unknown nodal heads, H. Shamir and Howard (1968) solved these equations using the Newton-Raphson method. After the nodal heads are computed, they can be substituted in Eq. 5-15 to compute the pipe flow rates.

5.2.3.4 Pipe Equations

The previous methods solve for the pipe flows, Q, or nodal heads, H, in a nonlinear solution scheme then use conservation of energy to determine the other set of unknowns. Haman and Brameller (1972) and Todini and Pilati (1987) devised a method to solve for Q and H simultaneously. They wrote the node equations (Eq. 5-11) with respect to the pipe flows and Eq. 5-14 for each pipe including both the pipe flows and the nodal heads. Although the number of equations (*nnode* + *npipe*) is larger than the other methods, the solution times and the convergence to the true solution are similar or better. In addition, the algorithm does not require defining loops that may be a time consuming task.

5.3 SOLUTION ALGORITHMS FOR STEADY FLOW

5.3.1 Solution of the Loop Equations

5.3.1.1 Hardy Cross Method (Single Loop Adjustment Algorithm)

The oldest and probably best known solution method for pipe networks is the Hardy Cross method that is found in most textbooks and taught in undergraduate hydraulics courses (Cross 1936). As noted in Section 5.2.3.1, the method solves the energy equations for loops and pseudo-loops for a loop flow correction. Although a set of loop equations must be solved for the system, this algorithm was developed for hand calculations and solves one loop at a time. One closed loop equation is written (Eq. 5-12) for each loop. For closed loops that contain only pipes, the loop equation for loop *LP* by Eq. 5-13 is:

$$F_{LP}(Q) = \sum_{l \in lloop} K_l \lfloor Q_l \rfloor^n \tag{5-16}$$

In this equation, the sign of Q_l is applied to the term and the absolute value is raised to the exponent *n*. The sign is based on the flow direction relative to loop *LP* and discussed in more detail below.

Since the flow rates that satisfy the set of loop equations are not known, a loop equation is expanded to a Taylor series truncated at the first order term or:

$$\begin{aligned} F_{LP}(Q^{(m)}) &\cong F_{LP}(Q^{(m-1)}) + \sum_{l \in lloop}(Q^{(m)} - Q^{(m-1)})\frac{dF_{LP}}{dQ_l} \\ &= \sum_{l \in lloop} K_l \lfloor Q^{(m-1)} \rfloor^n + \sum_{l \in lloop}(Q^{(m)} - Q^{(m-1)})\frac{dF_{LP}}{dQ_l} \end{aligned} \tag{5-17}$$

where $Q^{(m\text{-}1)}$ is the estimate of the flow at iteration m-1 and $\partial F_{LP} / \partial Q_l$ is the derivative of the LP^{th} loop equation with respect to the l^{th} pipe flow, Q_l. Defining $\Delta Q = Q^{(m)} - Q^{(m\text{-}1)}$ and substituting in Eq. 5-17:

$$F_{LP}(Q^{(m-1)}) + \Delta Q \sum_{l \in lloop} \left. \partial F_{LP} \middle/ \partial Q_l \right|_{Q^{(m-1)}} = 0 \Rightarrow \Delta Q = -\frac{F(Q^{(m-1)})}{\sum_{l \in lloop} \left. \partial F_{LP} \middle/ \partial Q_l \right|_{Q^{(m-1)}}} \tag{5-18}$$

The development of Eq. 5-18 is equivalent to the Newton-Raphson method except that ΔQ is computed rather than the updated flow $Q^{(m)}$. In addition, the Hardy Cross method simplifies the determination of the correction term by considering each loop independently rather than all loops simultaneously. Since all pipes in a loop will have the same correction, a single ΔQ is determined by Eq. 5-18. The numerator of Eq. 5-18 is computed from Eq. 5-16 with appropriate signs for flow directions. Standard convention is to define clockwise flow in each loop as positive. If $F(Q)$ equals zero, the equation has been satisfied.

The denominator is the sum of the absolute values of the derivative terms of Eq. 5-16 evaluated at $Q^{(m\text{-}1)}$. The individual gradient terms are:

$$\frac{\partial F}{\partial Q} = \frac{\partial (K(Q+\Delta Q)^n)}{\partial Q}\Big|_{Q^{(m-1)}} = n\,K\,Q^{n-1} = n\,h_L / Q \qquad (5\text{-}19)$$

For loop *LP*, Eq. 5-18 is then:

$$\Delta Q_{LP} = -\frac{\sum_{l\in lloop} K_l\, Q_l^n}{\sum_{l\in lloop} n\, K_l \lfloor Q_l \rfloor^{n-1}} = -\frac{\sum_{l\in lloop} K_l\, Q_l^n}{\sum_{l\in lloop} n\, K_l \lfloor Q_l \rfloor^{n} / Q_l} = -\frac{\sum_{l\in lloop} K_l\, Q_l^n}{\sum_{l\in lloop} n \left| h_{L,l} / Q_l \right|}$$

(5-20)

where the denominator becomes an absolute value because the signs of h_L and Q_l are the same.

A similar equation can be written for each loop in the network. Since a first-order Taylor series is used to approximate a nonlinear equation, a single set of corrections is likely insufficient to converge to the true flows and the process must be repeated until all loops equations are satisfied within a desired tolerance.

In summary, the Hardy Cross algorithm consists of the following steps.

1) Define loops and set $m = 0$. Assume an initial set of pipe flows that satisfy conservation of mass at all nodes. Note that the loop corrections will maintain conservation of mass after this initial step.
2) Update $m = m + 1$
3) Compute the sum of head losses around a loop by solving Eq. 5-16 for each loop substituting $Q^{(m\text{-}1)}$ for Q. This sum is the numerator of Eq. 5-20.
4) Compute the denominator of Eq. 5-20 for each loop. Note that the denominator is the sum of the absolute values of $n\ h_L/Q$ over the set of pipes in the loop, *lloop*.
5) Compute the loop correction, ΔQ_{LP}, by solving Eq. 5-20.
6) Repeat steps 3-5 for each loop.
7) Apply correction factors to all l pipes or:

$$Q_l^{(m)} = Q_l^{(m-1)} \pm \sum_{lp\,\in\, ncp(l)} \Delta Q_{lp}$$

   where $ncp(l)$ is the set of one or two loops in the network that contain pipe l.
8) Check if all ΔQ's are less than specified small tolerance. If so, stop. If not, go to step 2.

The equations above account for the flow direction change without modification. In the numerator, the appropriate sign is taken from the flow rate as defined by the initially assumed flow rate and relative to the loop being considered. Absolute values are always taken in the denominator. A negative flow relative to the initially defined loops denotes that the flow is in the opposite direction of the original assumption with the computed magnitude.

In step 7, the corrections are applied with similar logic. A positive correction implies a larger flow in the clockwise direction. Thus, ΔQ is added to pipes with assumed positive flow directions. If the assumed direction is counter-clockwise, the correction is subtracted from the $Q^{(m-1)}$. This convention is also applied if the actual flow directions is opposite of the assumed direction.

In the approach described above, the loop corrections are applied after all corrections have been computed. It is possible to be more sophisticated by, for example, applying the corrections as the method proceeds through an iteration. However, although the Hardy Cross method is acceptable for hand calculation, it is not efficient for or applied to large systems so these improvements are not considered here.

Example 5.4

Problem: Determine the flow rates in the pipes in the three loop network in Figure E5-4a and the nodal heads at all nodes using the Hardy Cross method and the Hazen-Williams equation. With the flow rates, compute the energy at node 5.

Solution: *Step 1*: m = 0. Following the procedure outlined above, the first step is to define a set of loops. Four loops are identified in Figure E5-4b. The head loss around the closed loops I, II and III is zero since the loops begin and end at the same node and the node has a unique total head (as in the parallel pipe analysis in Section 5.1.2). A pseudo-loop is defined between the two reservoirs and the difference in energy between the locations is 200 ft. The positive direction is defined in the clockwise direction for all loops.

Assuming the flow directions shown on Figure E5-4b, the loop equations are shown below. For Loop P (the pseudo-loop) starting at reservoir 1 and continuing along a path to reservoir 2:

$$Loop\ P:\ -K_1\lfloor Q_1\rfloor^{1.85} - K_2\lfloor Q_2\rfloor^{1.85} + K_9\lfloor Q_9\rfloor^{1.85} - (240 - 0.9376\,Q_P^2) =$$
$$H_{res.2} - H_{res.1} = 0 - 200 = -200$$

We have added the sign convention relative to the loop and the initially assumed flow directions. Pipe 1 is assumed to flow in the counter-clockwise direction relative to loop *P* and its term is given a negative sign. A positive flow in pipe 1 denotes that the flow direction is from the tank to node 1. If flow is assumed incorrectly and flow is actually from node 1 toward the tank in pipe 1, the sign is switched to a negative. In this case, pipe 1's head loss term in Loop *P* will be positive (the initially assumed negative sign times the negative sign from the flow term).

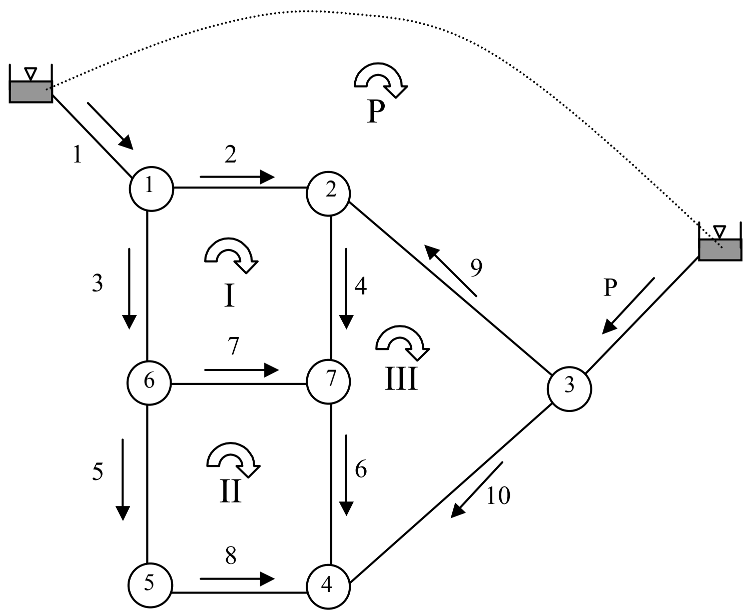


Figure E5-4b: Example hydraulic analysis pipe network with defined loops and assumed flow directions.

Pipe 9 is assumed to flow from node 3 to node 2 that is positive relative to loop *P* and its head loss term is positive. The direction of flow through the pump is also clockwise but the pump provides a head gain so it is given a negative sign. No sign convention is applied to the pump since it can only be non-negative, that is, a positive flow or zero.

For loop *III* beginning and ending at node 3:

$$+K_{10}\lfloor Q_{10}\rfloor^{1.85} - K_6\lfloor Q_6\rfloor^{1.85} - K_4\lfloor Q_4\rfloor^{1.85} - K_9\lfloor Q_9\rfloor^{1.85} = H_3 - H_3 = 0$$

Again, positive signs are given for flow moving clockwise relative to the loop. Thus, pipe 9 is positive relative to loop *P* but negative in loop *III*.

For loop *II* (beginning and ending at node 6):

$$+K_7\lfloor Q_7\rfloor^{1.85} + K_6\lfloor Q_6\rfloor^{1.85} - K_8\lfloor Q_8\rfloor^{1.85} - K_5\lfloor Q_5\rfloor^{1.85} = H_6 - H_6 = 0$$

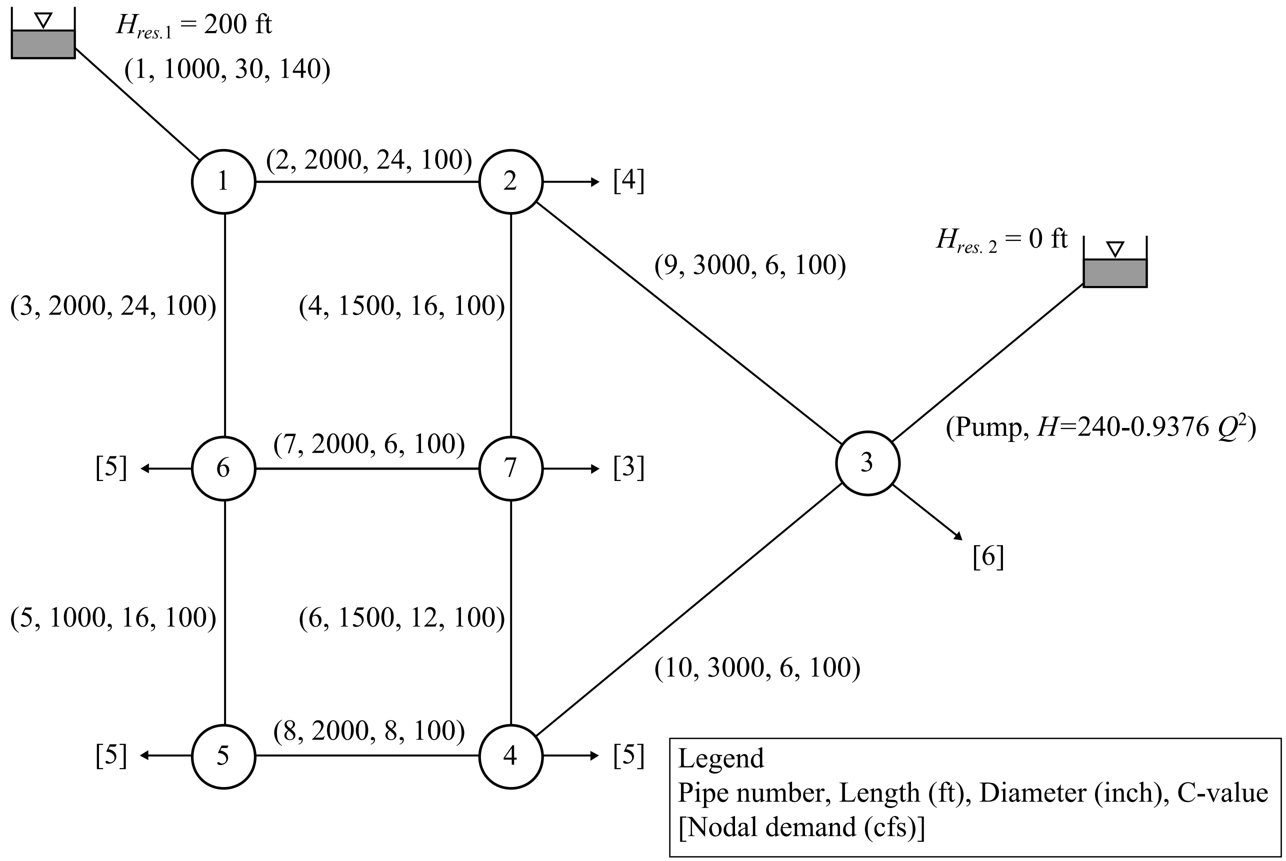


Figure E5-4a: Example hydraulic analysis pipe network.

Finally, loop *I* begins and ends at node 1:

$$+K_2 \lfloor Q_2 \rfloor^{1.85} + K_4 \lfloor Q_4 \rfloor^{1.85} - K_7 \lfloor Q_7 \rfloor^{1.85} - K_3 \lfloor Q_3 \rfloor^{1.85} = H_1 - H_1 = 0$$

Any pipe that appears in two loops has a negative sign in one loop equation and a positive in the other equation. This convention must also be considered when updating flows.

The *K* for each pipe is given in Table E5-4a using:

$$K = 4.73 \frac{L}{D^{4.87} \, C_{HW}^{1.85}}$$

with *Q* in cfs, *D* and *L* in ft. Table E5-4a also gives a set of pipe flow rates that satisfy conservation of mass. Initial pipe flows were determined for the sequence of nodes 1, 3, 2, 6, 7, and 5. The last node is then checked to confirm system mass balance. Since the total network demand is 28 cfs, the flows in pipe 1 and the pump must equal 28 cfs. To begin, the flow in pipe 1 is assumed to be 20 cfs. At a given node, all but one outflow pipe flows are assumed and the last value is computed by the node's mass balance equation.

Step 2: Set $m = m + 1 = 1$

Table E5-4a: Initial data and *K* coefficients for Example 5.4.

Pipe	**1**	**2**	**3**	**4**	**5**
Unode	Tank	1	1	2	6
Dnode	1	2	6	7	5
K	0.00584	0.0645	0.0645	0.349	0.233
Q(0)	20	9	11	6	5.5
h_L	1.49	3.76	5.45	9.60	5.45
nh_L/Q	0.138	0.774	0.918	2.96	1.83

Pipe	**6**	**7**	**8**	**9**	**10**	***P***
Unode	7	6	5	3	3	Tank
Dnode	4	7	4	2	4	3
K	1.42	55.2	13.6	82.8	82.8	
Q(0)	3.5	0.5	0.5	1	1	8
h_L	14.37	15.31	3.77	82.79	82.79	180
nh_L/Q	7.60	56.7	14.0	153.33	153.33	15.0

* Unode and Dnode are the initially assumed upstream and downstream nodes, respectively.

Step 3: Compute sum of the head losses around each loop using the values listed in Table E5-4a for the assumed flow conditions and the loop equations defined in Step 1. The sums are listed in Table E5-4b. Using the initial guess, the residual of loop *P's* energy equation, F_p, is:

$$F_P(Q^{(0)}) = -K_1 \lfloor Q_1 \rfloor^{1.85} - K_2 \lfloor Q_2 \rfloor^{1.85} + K_9 \lfloor Q_9 \rfloor^{1.85} - (240 - 0.9376\, Q_P^2) + 200$$
$$= -0.00584\,(20)^{1.85} - 0.0645\,(9)^{1.85} + 82.8\,(1)^{1.85} - (240 - 0.9376\,(8)^2) + 200$$
$$= -1.49 - 3.76 + 82.8 - 180. + 200 = 97.55$$

Step 4: The computed sums of absolute values of the derivatives are also listed in Table E5-4b. For loop P:

$$\sum_{l \in lloop} n \left| h_{L,l} / Q_l \right| = n \left(\frac{h_{L,1}}{Q_1} + \frac{h_{L,2}}{Q_2} + \frac{h_{L,9}}{Q_9} \right) + 2\,(0.9376\, Q_P) =$$
$$= 0.138 + 0.774 + 153.33 + 2\,(0.9376(8)) = 169.24$$

Note that last term is the derivative of the pump equation with respect to Q.

Step 5 and 6: Compute the correction for each loop using Eq. 5-20.

For loop P:

$$\Delta Q_P = -\frac{\sum K\, Q^{1.85}}{\sum \left| n\, h_L / Q \right|} = -\frac{97.55}{169.24} = -0.576$$

Table E5-4b gives the values for all loops.

Table E5-4b: Loop corrections for iteration 1 of Hardy Cross method example.

Loop	*P*	*I*	*II*	*III*
$\Sigma\, h_L$	97.55	-7.40	20.46	-23.97
$\Sigma\, n\, \lvert h_L/Q \rvert$	169.25	61.37	80.12	317.23
ΔQ_l	-0.576	0.121	-0.255	0.076

Step 7: The loop corrections are applied to each pipe as follows. For pipe 1, the pseudo-loop's correction is applied with a negative sign since the correction is in the clockwise direction and pipe 1 is assumed to flow in the counter-clockwise direction or:

$$Q_1^{(1)} = Q_1^{(0)} - \Delta Q_P = 20 - (-0.576) = 20.576$$

Corrections for loops P and I are applied to pipe 2 since it is located in both loops. Loop P's correction is negative since pipe 2 flow is assumed to flow in the counter-clockwise direction for that loop (negative) and the loop I correction is positive since pipe 2's flow is clockwise relative to that loop.

$$Q_2^{(1)} = Q_2^{(0)} - \Delta Q_P + \Delta Q_I = 9 - (-0.576) + 0.121 = 9.697$$

Similarly for the other pipes:

$$Q_3^{(1)} = Q_3^{(0)} - \Delta Q_I = 11 - 0.121 = 10.879$$
$$Q_4^{(1)} = Q_4^{(0)} + \Delta Q_I - \Delta Q_{III} = 6 + 0.121 - 0.076 = 6.045$$
$$Q_5^{(1)} = Q_5^{(0)} - \Delta Q_{II} = 5.5 - (-0.255) = 5.755$$
$$Q_6^{(1)} = Q_6^{(0)} + \Delta Q_{II} - \Delta Q_{III} = 3.5 + (-0.255) - 0.076 = 3.169$$
$$Q_7^{(1)} = Q_7^{(0)} - \Delta Q_I + \Delta Q_{II} = 0.5 - 0.121 + (-0.255) = 0.124$$
$$Q_8^{(1)} = Q_8^{(0)} - \Delta Q_{II} = 0.5 - (-0.255) = 0.755$$
$$Q_9^{(1)} = Q_9^{(0)} + \Delta Q_P - \Delta Q_{III} = 1 + (-0.576) - 0.076 = 0.348$$
$$Q_{10}^{(1)} = Q_{10}^{(0)} + \Delta Q_{III} = 1 + 0.076 = 1.076$$
$$Q_P^{(1)} = Q_P^{(0)} + \Delta Q_P = 8 + (-0.576) = 7.424$$

Nodal flow balances continue to be conserved and can be verified. It is worthwhile to compute those balances to check if a computational or sign error has been introduced. In this case, pipe 10 is rounded down to preserve the mass balance for node 10.

Step 8: The maximum correction is 0.576 cfs so iterations continue. Go to step 2.

To provide some insight into the sign convention, consider pipe 1 during the first iteration. If we had assumed that pipe 1's flow was from node 1 to the tank and the flow was exiting the tank, the sign on Q_1 would be negative (i.e., Q_1 = -20). In calculating the correction, the denominator of ΔQ_P would be the same since the absolute value of the individual terms are summed. The resulting numerator would also be the same because a different sign would be applied to the pipe 1 term or:

$$F_P(Q^{(0)}) = +K_1 \lfloor Q_1 \rfloor^{1.85} - K_2 \lfloor Q_2 \rfloor^{1.85} + K_9 \lfloor Q_9 \rfloor^{1.85} - (240 - 0.9376\, Q_P^2) + 200$$
$$= +0.00584\,\mathbf{(-1)}\,(20)^{1.85} - 0.0645\,(9)^{1.85} + 82.8\,(1)^{1.85} - (240 - 0.9376\,(8)^2) + 200$$
$$= -1.49 - 3.76 + 82.8 - 180. + 200 = 97.55$$

Pipe 1 would be positive since the assumed flow from node 1 to the tank is a clockwise flow relative to loop *P*. However, the actual flow would carry a negative sign (shown in parenthesis) since flow was opposite of the assumed direction. Thus, the numerator and correction term would not change.

Based on the flow direction assumption, the flow is in the positive (clockwise) direction for loop *P* so the correction would be added to Q_1 or:

$$Q_1^{(1)} = Q_1^{(0)} + \Delta Q_P = -20 + (-0.576) = -20.576$$

Thus, the magnitude for the next iteration would be same as above and the negative sign would denote that the flow was in the opposite direction of the assumption (node 1 to tank).

A number of iterations are required for the Hardy Cross method to converge to the solution. Flow values and the loop corrections are given in Table E5-4c and d, respectively, for 11 additional iterations until the largest loop correction is less than 0.02 cfs. Values in the tables are computed without rounding in the spreadsheet that performed the calculations.

The total head at node 5 can be computed by beginning a path at either reservoir 1 or 2. Starting at reservoir 1 with a head of 200 ft, a path to node 5 is pipes 1, 3, and 5. Signs on the energy loss terms are based on flow directions in the path. Since flow goes from the tank to node 1, energy is lost as the water travels through pipe 1. Similarly, energy is lost as flow moves from node 1 to 6 in pipe 3 and from node 6 to 5 in pipe 5. The overall energy equation is then:

$$H_{res.1} - K_1 Q_1^{1.85} - K_3 Q_3^{1.85} - K_5 Q_5^{1.85} = H_5$$
$$= 200 - 0.00584\ (21.27)^{1.85} - 0.0645\ (11.40)^{1.85} - 0.233\ (6.06)^{1.85} = 185.98\ ft$$

Table E5-4c: Pipe flow values for Hardy Cross iterations.

	Pipe Flow (cfs)										
m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**P**
2	20.80	9.58	11.22	5.96	5.74	3.45	0.49	0.74	0.38	0.82	7.20
3	21.01	9.89	11.11	6.16	5.91	3.37	0.20	0.91	0.27	0.72	6.99
4	21.12	9.80	11.31	6.05	5.89	3.47	0.43	0.89	0.24	0.64	6.88
5	21.19	9.93	11.26	6.14	5.99	3.41	0.27	0.99	0.21	0.60	6.81
6	21.23	9.86	11.36	6.06	5.97	3.45	0.39	0.97	0.20	0.57	6.78
7	21.25	9.91	11.34	6.10	6.03	3.41	0.31	1.03	0.19	0.56	6.75
8	21.26	9.86	11.39	6.05	6.02	3.43	0.38	1.02	0.19	0.56	6.75
9	21.26	9.88	11.38	6.07	6.05	3.40	0.33	1.05	0.19	0.55	6.74
10	21.27	9.86	11.41	6.04	6.04	3.41	0.37	1.04	0.19	0.55	6.74
11	21.27	9.87	11.40	6.05	6.06	3.40	0.35	1.06	0.19	0.55	6.73

Node 5's head can also be computed starting at reservoir 2. In this path, energy is gained as water is lifted by the pump and lost as water moves from node 3 to 4 in pipe 10. The energy at node 5 is greater than at node 4 since water is flowing from node 5 to 4 in pipe 8. The head loss in pipe 8 therefore must be added in the path equation from reservoir 2 to node 5 or:

$$H_{res.2} + (240 - 0.9376 Q_P^2) - K_{10} Q_{10}^{1.85} + K_8 Q_8^{1.85} = H_5$$
$$= 0 + (240 - 0.9376 (6.73)^2) - 82.8\ (0.55)^{1.85} + 13.6\ (1.06)^{1.85} = 185.28\ ft$$

Slight differences result from the two paths since the Hardy Cross iterations were stopped before full convergence.

Table E5-4d: Loop corrections for Hardy Cross iterations.

Iteration	Loop *P*	Loop *I*	Loop *II*	Loop *III*
2	-0.227	-0.342	0.020	-0.260
3	-0.203	0.108	-0.177	-0.093
4	-0.111	-0.200	0.026	-0.081
5	-0.073	0.055	-0.104	-0.040
6	-0.035	-0.106	0.019	-0.027
7	-0.022	0.026	-0.056	-0.011
8	-0.008	-0.054	0.010	-0.009
9	-0.007	0.012	-0.029	-0.003
10	-0.002	-0.027	0.006	-0.003
11	-0.003	0.006	-0.015	-0.001

5.3.1.2 Simultaneous Loop Equation Solution (Simultaneous Loop Flow Adjustment Algorithm)

In the Hardy Cross method, each loop correction is determined independently of the other loops. As seen in Figure E5-4b, several loops may have common pipes so corrections to those loops will impact energy losses around more than one loop. Epp and Fowler (1970) developed a more efficient approach by simultaneously computing corrections for all loops. As in the Hardy Cross method, an initial solution that satisfies continuity at all nodes is required. For a simultaneous loop equation solution, Eq. 5-16 for loop *LP* becomes:

$$F_{LP}(Q) = \sum_{l \in lloop} K_l \left[(Q_l^{(m-1)} + \sum_{lp \in ncp(l)} \Delta Q_{lp}) \right]^n = 0 \qquad (5\text{-}21)$$

where *ncp(l)* is the set of one or two loops in the network that contain pipe *l* (e.g., loops *P* and *I* for pipe 2 in Example 5.4). The sign convention on the pipe flow relative to the loop is the same as for the Hardy Cross method. The Newton-Raphson method is then used to solve Eq. 5-21 for the ΔQ's. A system of linear equations must now be iteratively solved rather than the single equations of the Hardy Cross method.

A first order Taylor series approximation of Eq. 5-21 for loop *LP* is:

$$\left(\sum_{l \in lloop} \frac{\partial F_{LP}}{\partial (\Delta Q_{lp})}\bigg|_{Q^{(m-1)}} \right) \Delta Q_{LP} + \sum_{lp \in ncl(LP)} \left(\frac{\partial F_{LP}}{\partial (\Delta Q_{lp})}\bigg|_{Q^{(m-1)}} \Delta Q_{lp} \right) = -F_{LP}(Q^{(m-1)}) \qquad (5\text{-}22)$$

where *ncl(LP)* is the set of loops that have a common pipe with loop *LP* (e.g., loops *I* and *III* for loop *II* in Example 5.4). In vector form for all loops simultaneously Eq. 5-22 can be written as:

$$\mathbf{J_L}\ \Delta\mathbf{Q} = -\mathbf{F}(\mathbf{Q}^{(m-1)}) \qquad (5\text{-}23)$$

where $\mathbf{Q}^{(m-1)}$ is the vector of *npipe* pipe flow, $\Delta\mathbf{Q}$ is the [1 x (*nloop*+*nploop)*] vector of loop flow corrections and $\mathbf{F(Q}^{(m-1)})$ is the [1 x (*nloop*+*nploop)*] vector of residuals of the loop conservation of energy equations (Eq. 5-14) evaluated at $\mathbf{Q}^{(m-1)}$. Residuals are the values of the right hand side at the trial values of **Q**. The objective is for all of those terms to equal zero such that all loop equations are satisfied.

$\mathbf{J_L}$ equals $\partial\mathbf{F}/\partial(\Delta\mathbf{Q})$ and is the Jacobian matrix of first derivatives of the loop equations evaluated at $\mathbf{Q}^{(m-1)}$. $\mathbf{J_L}$ is square [(*nloop*+*nploop)* x (*nloop*+*nploop)*], symmetric and positive definite. The rows in $\mathbf{J_L}$ correspond to the loop equations and the columns are related to the loop corrections.

The LP^{th} diagonal term of the Jacobian is the sum of the first derivatives of the *lloop* pipes in loop *LP* or the summation in the first term in Eq. 5-22. This term is also equivalent to the denominator of Eq. 5-20.

The difference between the simultaneous loop method and the Hardy Cross method is that some of the off-diagonal terms are non-zero. The Jacobian term in row *LP* (loop *LP*'s conservation of energy equation) and column *lp* (common loop) corresponds to the gradient term for the pipe that is common to loops *LP* and *lp*. These terms are equal to zero if the loops do not have common pipes. If the loops have common pipes, these terms are the negative of the sum of the absolute values of $-|\partial F/\partial Q| = -|n\ h_L\ /Q|$ for pipes that appear in loop *LP* and *lp*. The negative sign results because the flow direction in loop *lp* is opposite to loop *LP*. An example of forming Eq. 5-23 is given in Example 5.5.

Once the matrices are formed, Eq. 5-23 can be solved by any linear equation solver for $\Delta\mathbf{Q}$. The pipe flows are updated by the loop corrections as in the Hardy Cross method (i.e., $\mathbf{Q}^{(m)} = \mathbf{Q}^{(m-1)} +/- \Delta\mathbf{Q}$). The solution algorithm is the same as the Hardy Cross method except steps 5 and 6 are reduced to a single step and all corrections are computed simultaneously. Since the equations are nonlinear, several iterations may be necessary to converge to the solution, like the Hardy Cross method. To end the algorithm, one of several stopping criteria can be applied: (1) number of iterations allowed or (2) the magnitude of the change in loop flow rates, $\Delta\mathbf{Q}$.

As in the Hardy Cross method, if a pipe's flow direction changes from the assumed value, the signs for that pipe head loss terms are switched for all loops containing the pipe during the next iteration in the loop equations using the sign convention noted above. The direction change does not alter the coefficient matrix, $\mathbf{J_L}$, or the signs on the individual terms. The diagonal terms are the sum of the absolute values of the gradients and the off-diagonal terms are always negative.

Example 5.5

Problem: Determine the flow rates in the pipes in the three loop network in Example 5.4 and the nodal heads at all nodes using the simultaneous loop method and the Hazen-Williams equation.

Solution: The Example 5.4 starting point is used again in this example. Equation 5-23 is solved to provide the simultaneous loop corrections. For the example network, the four loop equations were developed in Example 5.4 for loops *P*, *I*, *II*, and *III* as:

$$P : -K_1 Q_1^{1.85} - K_2 Q_2^{1.85} + K_9 Q_9^{1.85} - (240 - 0.9376 Q_P^2) = H_{res2} - H_{res1} = -200$$

$$I : +K_2 Q_2^{1.85} + K_4 Q_4^{1.85} - K_7 Q_7^{1.85} - K_3 Q_3^{1.85} = H_1 - H_1 = 0$$

$$II : +K_7 Q_7^{1.85} + K_6 Q_6^{1.85} - K_8 Q_8^{1.85} - K_5 Q_5^{1.85} = H_6 - H_6 = 0$$

$$III : +K_{10} Q_{10}^{1.85} - K_6 Q_6^{1.85} - K_4 Q_4^{1.85} - K_9 Q_9^{1.85} = H_3 - H_3 = 0$$

The coefficient matrix terms are the gradients of the loop equations with respect to each loop flow correction or:

$$\mathbf{J_L} = \begin{bmatrix} \dfrac{\partial F_P}{\partial(\Delta Q_P)} & \dfrac{\partial F_P}{\partial(\Delta Q_I)} & \dfrac{\partial F_P}{\partial(\Delta Q_{II})} & \dfrac{\partial F_P}{\partial(\Delta Q_{III})} \\ \dfrac{\partial F_I}{\partial(\Delta Q_P)} & \dfrac{\partial F_I}{\partial(\Delta Q_I)} & \dfrac{\partial F_I}{\partial(\Delta Q_{II})} & \dfrac{\partial F_I}{\partial(\Delta Q_{III})} \\ \dfrac{\partial F_{II}}{\partial(\Delta Q_P)} & \dfrac{\partial F_{II}}{\partial(\Delta Q_I)} & \dfrac{\partial F_{II}}{\partial(\Delta Q_{II})} & \dfrac{\partial F_{II}}{\partial(\Delta Q_{III})} \\ \dfrac{\partial F_{III}}{\partial(\Delta Q_P)} & \dfrac{\partial F_{III}}{\partial(\Delta Q_I)} & \dfrac{\partial F_{III}}{\partial(\Delta Q_{II})} & \dfrac{\partial F_{III}}{\partial(\Delta Q_{III})} \end{bmatrix}$$

The diagonal terms are identical to the denominator of the Hardy Cross correction:

$$\frac{\partial F_{LP}}{\partial(\Delta Q_{LP})} = \sum_{l \in lloop} n \left| \frac{h_{L,l}}{Q_l} \right|$$

where *lloop* is the number of pipes in loop *LP*. From the values in Table E5-4a and b, a symmetric coefficient matrix can be filled. The rows 1-4 correspond to loops *P*, *I*, *II*, and *III*. As shows in step 4 of Example 5.4:

$$\frac{\partial F_{LP}}{\partial(\Delta Q_{LP})} = \sum_{l \in lloop} n\left|\frac{h_{L,l}}{Q_l}\right| = n\left(\frac{h_{L,1}}{Q_1} + \frac{h_{L,2}}{Q_2} + \frac{h_{L,9}}{Q_9}\right) + 2\,(0.9376\,Q_P) =$$
$$= 0.138 + 0.774 + 153.33 + 2\,(0.9376)\,(8) = 169.24$$

The off-diagonal terms are the gradients for pipes that appear in loop *LP* and another loop, *lp* or:

$$\frac{\partial F_{LP}}{\partial(\Delta Q_{lp})} = -\sum_{l \in ncpipe(LP,lp)} n\left|\frac{h_{L,l}}{Q_l}\right| = \frac{\partial F_{lp}}{\partial(\Delta Q_{LP})}$$

where *ncpipe* (*LP,lp*) is the set of pipes that are common to loops *LP* and *lp* (e.g., pipe 2 for loops *I* and *P*). The value in row 1 (corresponding to loop equation *P*) and column 2 (corresponding to loop *I*) is the gradient of the only common pipe 2 with respect to the flow correction in loop *I* or:

$$\frac{\partial F_P}{\partial(\Delta Q_I)} = -\sum_{l \in ncpipe(P,I)} n\left|\frac{h_{L,l}}{Q_l}\right| = -n\left|\frac{h_{L,2}}{Q_2}\right| = -1.852\left|\frac{3.76}{9}\right| = -0.774$$

As noted above, this gradient is also $\partial F_I / \partial(\Delta Q_P)$.

The remaining diagonal terms are given in Table E5-4b and the off-diagonals correspond to pipes common to two loops (Table E5-4a). Pipe 2 appears in loops *P* and *I*, pipe 9 appears in *P* and *II*, pipe 4 is located in *I* and *II* and pipe 6 is in loops *II* and *III*. No pipe is common to loops *P* and *II* so the coefficients are zero in those locations (column 3-row 1 and column 1-row 3). The $\mathbf{J_L}$ matrix is then:

$$\mathbf{J_L} = \begin{bmatrix} 169.25 & -0.774 & 0 & -153.33 \\ -0.774 & 61.37 & -56.71 & -2.96 \\ 0 & -56.71 & 80.12 & -7.60 \\ -153.33 & -2.96 & -7.60 & 317.23 \end{bmatrix}$$

The right hand side of the equation is the computed value of the equation with the current flow estimates that were computed for the Hardy Cross method and listed in the second row of Table E5-4b.

$$\mathbf{F}(\mathbf{Q}^{(0)})^T = [97.55.\ -7.40,\ 20.46,\ -23.97]$$

For example, row 3 corresponds to loop *II*:

$$F_{II} = K_7\, Q_7^{1.85} + K_6\, Q_6^{1.85} - K_8\, Q_8^{1.85} - K_5\, Q_5^{1.85} =$$
$$= 55.2\,(0.5)^{1.85} + 1.42\,(3.5)^{1.85} - 13.6\,(0.5)^{1.85} - 0.233\,(5.5)^{1.85} =$$
$$= 15.31 + 14.37 - 3.77 - 5.45 = 20.46$$

The system of linear equations, $\mathbf{J_L}\,\Delta\mathbf{Q} = -\mathbf{F}$, is solved for the unknown flow corrections.

$$\mathbf{J_L}\ \Delta\mathbf{Q} = \begin{bmatrix} 169.25 & -0.774 & 0 & -153.33 \\ -0.774 & 61.37 & -56.71 & -2.96 \\ 0 & -56.71 & 80.12 & -7.60 \\ -153.33 & -2.96 & -7.60 & 317.23 \end{bmatrix} \begin{bmatrix} \Delta Q_P \\ \Delta Q_I \\ \Delta Q_{II} \\ \Delta Q_{III} \end{bmatrix}$$
$$= -\mathbf{F} = \begin{bmatrix} -97.55 \\ +7.40 \\ -20.46 \\ +23.97 \end{bmatrix}$$

Resulting in:

$$\Delta\mathbf{Q}^{T} = [\ \Delta Q_P, \Delta Q_I, \Delta Q_{II}, \Delta Q_{III}\ ]^{T} = [-0.942, -0.525, -0.665, -0.404]^{T}$$

The corrections are applied using the equations shown in Example 5.4, step 7. For example, the flow rates in pipes 1 and 2 become:

$$Q_2^{(1)} = Q_2^{(0)} - \Delta Q_P + \Delta Q_I = 9 - (-0.942) + (-0.525) = 9.416$$

The loop flow corrections are still large so additional iterations are needed to converge to the solution. Since flow corrections are made on all loops simultaneously, this method converges in four iterations (to an absolute change in Q of 0.001 cfs) compared to 11 for the Hardy Cross method (with a larger tolerance). The results are summarized in Tables E5-5a and b.

Since the flow rates from the Hardy Cross and simultaneous loop methods are the same the nodal heads will be the same for both results.

5.3.2 Solution of the Node-Loop Equations (Flow Adjustment Algorithm)

Using the loop equations to represent conservation of energy, Wood and Charles (1972) developed the linear theory (Flow Adjustment) method by coupling the loop equations with the node equations. Wood and Rayes (1981) later showed that a modified linear theory (presented here) exhibits superior convergence characteristics compared with the original linear theory method.

Table E5-5a: Pipe flow values for the simultaneous loop correction method.

Iter.	Pipe Flow (cfs)										
m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	***P***
2	20.94	9.42	11.53	5.88	6.17	3.24	0.36	1.17	0.46	0.60	7.06
3	21.21	9.79	11.43	6.04	6.07	3.39	0.36	1.07	0.25	0.54	6.79
4	21.27	9.84	11.43	6.03	6.07	3.39	0.36	1.07	0.19	0.54	6.73
5	21.27	9.85	11.43	6.03	6.07	3.39	0.36	1.07	0.19	0.54	6.73

Table E5-5b: Loop corrections for simultaneous loop correction iterations.

Iteration	Loop *P*	Loop *I*	Loop *II*	Loop *III*
1	-0.942	-0.525	0.053	-0.404
2	-0.271	0.100	0.097	0.060
3	-0.053	0.001	0.002	0.004
4	-0.005	0.	0.	0.001

In the modified method, rather than solve for loop corrections and be required to provide a feasible initial solution, conservation of energy around a loop (Eq. 5-16) is written directly in terms of the pipe flow rates or for a closed loop:

$$F(Q) = \sum_{l \in lloop} K_l \lfloor Q_l \rfloor^n = 0 \tag{5-24}$$

However, the number of unknown pipe flows is equal to the number of pipes (*np*) but only *nloop* + *nploop* equations of the form of Eq. 5-16 are available. Therefore, these equations are coupled with the nodal conservation of mass equations (Eq. 5-11):

$$\sum_{l \in J_{in}} Q_l - \sum_{l \in J_{out}} Q_l = q$$

With conservation of mass, the number of equations is *nnode* node equations plus *nloop* closed loop equations and *nploop* pseudo-loop equations or a total of *np* equations written in terms of the *np* unknown pipe flow rates.

These nonlinear equations are also solved iteratively by applying the Newton-Raphson method. Taking a Taylor series expansion of a loop equation results in:

$$\sum_{l \in lloop} \left(\left. \frac{dF_{LP}}{dQ_l} \right|_{Q^{(m-1)}} (Q_l^{(m)} - Q_l^{(m-1)}) \right) = -F_{LP}(Q^{(m-1)})$$

where F is the loop equation (Eq. 5-16) and $Q^{(m-1)}$ are the known pipe flows for the previous iteration and $Q^{(m)}$ is the unknown flow rates at iteration m. This equation can be rearranged with the known terms on the right hand side as:

$$\sum_{l \in lloop} \left(\left. \frac{dF}{dQ_l} \right|_{Q^{(m-1)}} Q_l^{(m)} \right) = -F(Q^{(m-1)}) + \sum_{l \in lloop} \left(\left. \frac{dF}{dQ_l} \right|_{Q^{(m-1)}} Q_l^{(m-1)} \right) \qquad (5\text{-}25)$$

A similar relationship can be written for conservation of mass. It is linear with respect to the unknowns, $\mathbf{Q}^{(m)}$ since the gradient terms and the functions F can be evaluated at $\mathbf{Q}^{(m-1)}$. Eq. 5-25 can be written in matrix form as:

$$\mathbf{J_{NL}}\,\mathbf{Q}^{(m)} = \mathbf{F_{NL}} = -\,\mathbf{F} + \mathbf{J_{NL}}\,\mathbf{Q}^{(m-1)} \qquad (5\text{-}26)$$

where $\mathbf{J_{NL}}$ is the Jacobian of the node-loop equations and $\mathbf{F_{NL}}$ is the vector of functions of known values from the previous iteration. $\mathbf{J_{NL}}$ and $\mathbf{F_{NL}}$ vary between iterations as $\mathbf{Q}$ moves toward the solution. $\mathbf{F}$ is the vector of residuals computed by substituting $\mathbf{Q}^{(m-1)}$ in the node-loop equations.

The rows in $\mathbf{J_{NL}}$ correspond to the conservation of mass and energy equations and the columns relate to the unknown pipe flow rates. For the conservation of mass equation for node i, the terms in the corresponding row in $\mathbf{J_{NL}}$ will be 0 if the pipe is not connected to node i, +1 if the pipe is carrying flow to the node (i.e., the pipe is in the set $J_{in,i}$, e.g., pipes 6 and 8 for node 4 in Example 5.4) and –1 if the pipe is in set $J_{out,i}$ and carries water from node i (e.g., pipes 2 and 3 for node 1 in Ex. 5.4). For the conservation of energy equations, the gradient terms are the same as the Hardy Cross terms (Eq. 5-21) (i.e., $|n\,h_L/Q| = |nKQ^{\,n-1}|$), if the pipe appears in the loop, and zero, otherwise. The full term becomes more complex when a pump appears in the loop (Example 5.6).

$\mathbf{J_{NL}}$ and $\mathbf{F_{NL}}$ are evaluated at $\mathbf{Q}^{(m-1)}$ and Eqs. 5-26 are solved for the new pipe flows, $\mathbf{Q}^{(m)}$. This iterative process continues until a defined stopping criteria is met, such as when the absolute or percentage difference between two iterations flows, $\mathbf{Q}^{(m)}$ and $\mathbf{Q}^{(m-1)}$, is less than a tolerance for all pipes or a limiting number iterations are completed. Since conservation of mass is solved as part of Eq. 5-26, the initial solution does not have to satisfy this condition.

Example 5.6

Problem: Determine the flow rates in the pipes in the three loop network in Example 5.4 and the nodal heads at all nodes using the modified linear theory method and the Hazen-Williams equation. Use the same starting point as Example 5.4.

Solution: The node-loop equations consist of the node equations written with respect to the pipe flow (Eq. 5-11) and the loop equations (Eq. 5-16).

$$\begin{aligned}
Node\,1 &: \lfloor Q_1 \rfloor - \lfloor Q_2 \rfloor - \lfloor Q_3 \rfloor = q_1 = 0 \\
Node\,2 &: \lfloor Q_2 \rfloor - \lfloor Q_4 \rfloor + \lfloor Q_9 \rfloor = q_2 = 4 \\
Node\,3 &: -\lfloor Q_9 \rfloor - \lfloor Q_{10} \rfloor + Q_P = q_3 = 6 \\
Node\,4 &: \lfloor Q_6 \rfloor + \lfloor Q_8 \rfloor + \lfloor Q_{10} \rfloor = q_4 = 5 \\
Node\,5 &: \lfloor Q_5 \rfloor - \lfloor Q_8 \rfloor = q_5 = 5 \\
Node\,6 &: \lfloor Q_3 \rfloor - \lfloor Q_5 \rfloor - \lfloor Q_7 \rfloor = q_6 = 5 \\
Node\,7 &: \lfloor Q_4 \rfloor + \lfloor Q_7 \rfloor - \lfloor Q_6 \rfloor = q_7 = 3
\end{aligned}$$

$$Loop\ P:\ -K_1 \lfloor Q_1 \rfloor^{1.85} - K_2 \lfloor Q_2 \rfloor^{1.85} + K_9 \lfloor Q_9 \rfloor^{1.85} - (240 - 0.9376 Q_P^2) = $$
$$= H_{res.2} - H_{res.1} = 0 - 200$$
$$Loop\ I:\ +K_2 \lfloor Q_2 \rfloor^{1.85} + K_4 \lfloor Q_4 \rfloor^{1.85} - K_7 \lfloor Q_7 \rfloor^{1.85} - K_3 \lfloor Q_3 \rfloor^{1.85} = H_1 - H_1 = 0$$
$$Loop\ II:\ +K_7 \lfloor Q_7 \rfloor^{1.85} + K_6 \lfloor Q_6 \rfloor^{1.85} - K_8 \lfloor Q_8 \rfloor^{1.85} - K_5 \lfloor Q_5 \rfloor^{1.85} = H_6 - H_6 = 0$$
$$Loop\ III:\ +K_{10} \lfloor Q_{10} \rfloor^{1.85} - K_6 \lfloor Q_6 \rfloor^{1.85} - K_4 \lfloor Q_4 \rfloor^{1.85} - K_9 \lfloor Q_9 \rfloor^{1.85} = 0$$

These 11 equations can be solved for the 10 pipe flows and 1 pump flow. An arbitrary positive flow direction has been assigned to each pipe that is consistently applied in the conservation of mass and energy equations. For example, pipe 2 is positive when water flow from node 1 to node 2. Thus, it is an outflow from node 1 and is given a negative sign in that conservation of mass equation. It is an inflow to node 2 and given a positive sign in that node's mass balance equation. In addition, choosing clockwise as positive in all conservation of energy equations, flow from node 1 to 2 is counterclockwise (negative) in loop *P* and clockwise (positive) in loop *I*. Using our convention for taking the sign of the flow rate, flows that are opposite of the assumed direction become negative and change the signs on the terms. For example, if Q_2 became –2 with $n = 1.85$ then:

$$\lfloor Q_2 \rfloor^{1.85} = -(|Q_2|)^{1.85} = -|2|^{1.85} = -3.61.$$

Thus, negative flows are possible and distinguish the proper magnitude and that the assumed direction was incorrect.

As noted above, the columns in the coefficient matrix, $\mathbf{J_{NL}}$, for these equations correspond to pipes and the rows correspond to the nodes plus loop equations. The first seven (*nnode*) rows of the right hand side vector are nodal demands (Eq. 5-11). For node 6, the non-zero Jacobian terms correspond to pipes 3, 5 and 7. For that node, pipe 3 is in inflow pipe and 5 and 7 outflow pipes so the gradients for these pipes are:

$$\partial F_{N,6} \big/ \partial Q_3 = 1; \quad \partial F_{N,6} \big/ \partial Q_5 = -1; \quad \partial F_{N,6} \big/ \partial Q_7 = -1$$

where $F_{N,6}$ denotes the node equation for node 6. The sixth row in $\mathbf{J_{NL}}$ is:

$$\mathbf{J}_{\mathbf{NL},\text{ row }6} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The loop equations are linearized by a Taylor series expansion and the terms in the gradient matrix, $\partial F_{LP} / \partial Q_l$, are the derivatives of loop equation for loop LP with respect to flow in pipe l or $n\, h_{L,l} / Q_l$ for the pipes in the loop and zero otherwise. The sign for the term relates if the assumed flow is clockwise (+) or counter-clockwise (-) relative to the loop. Therefore, values of the last four rows of $\mathbf{J_{NL}}$ corresponding to the loop equations come directly from the last row of Table E5-4a. Based on the assumed flow directions, the signs are: (Loop P = [-1, -2, 9, -Pump]), (Loop I = [2, -3, 4, -7]), (Loop II = [-5, 6, 7, -8]), and (Loop III = [-4, -6, -9, 10]).

For the initial point used in the Hardy Cross method, $\mathbf{J_{NL}}$ is:

$$\mathbf{J_{NL}} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -0.138 & -0.774 & 0 & 0 & 0 & 0 & 0 & 0 & 153.33 & 0 & 15.0 \\ 0 & 0.774 & -0.918 & 2.96 & 0 & 0 & -56.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.83 & 7.60 & 56.7 & -13.97 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.96 & 0 & -7.60 & 0 & 0 & -153.33 & 153.33 & 0 \end{bmatrix}$$

The right hand side of Eq. 5-24 for the node equations is equal to the nodal demand, q, demonstrating that the linearization of the node results does not alter the equation. For node 6 with the assumed flows the RHS is:

$$-F_{N,6}(Q^{(0)}) + \sum_{l \in J_{in}, J_{out}} \left.\frac{\partial F_{N,6}}{\partial Q_l}\right|_{Q^{(0)}} Q_l^{(0)}$$

$$= -(Q_3 - Q_5 - Q_7 - q_6) + \frac{\partial F_{N,6}}{\partial Q_3} Q_3^{(0)} + \frac{\partial F_{N,6}}{\partial Q_5} Q_5^{(0)} + \frac{\partial F_{N,6}}{\partial Q_7} Q_7^{(0)}$$

$$= -(11 - 5.5 - 0.5 - 5) + (+1)(11) + (-1)(5.5) + (-1)(0.5) = 5$$

The reader can confirm that an unbalanced assumption of pipe flows gives the same results (e.g., $Q_3 = +10$, $Q_5 = -5.5$ and $Q_7 = -0.5$).

The last four rows are more than the deviations in the energy balances (i.e., $-F(Q)$) as in the simultaneous loop solution. The additional terms are the gradients of the loop equations (the last four rows of $\mathbf{J_{NL}}$) times the present flow estimates ($Q^{(m-1)}$). For loop *I* that contains only pipes, the right hand side is:

$$-F(Q^{m-1}) + \left.\frac{\partial F}{\partial Q}\right|_{Q^{m-1}} Q^{m-1} = -\sum h_L + \sum n\,(h_L / Q^{(0)})\,Q^{(0)}$$

$$= -\sum h_L + \sum n\,h_L = \sum (n-1)\,h_L \quad = 0.85\,(-7.40) = -6.29$$

The resulting relationship is different for loop *P* that contains a pump.

$$-F(Q^{m-1}) + \left.\frac{\partial F}{\partial Q}\right|_{Q^{m-1}} Q^{m-1}$$

$$= -\sum (h_L + h_P - \Delta E) + \sum n\,(h_L / Q^{(0)})\,Q^{(0)} + (2\,(-0.9376)\,Q_P^{(0)})Q_P^{(0)}$$

$$= -(97.55) + 1.85\,(77.55) + (2(-0.9376)\,8)\,8 = 165.93$$

The full loop equation is included in the first term including the energy difference for pseudo-loops. The gradient of the energy difference is zero so it does not appear in the second term.

For the first iteration, the resulting right hand side is:

$$\mathbf{F_{NL}}(\mathbf{Q}^{(0)}) = \begin{bmatrix} 0 \\ 4 \\ 6 \\ 5 \\ 5 \\ 5 \\ 3 \\ -(97.55) + 1.85\,(-1.49 - 3.76 + 82.8) + 120.01 \\ -(-7.40) + 1.85\,(3.76 + 9.60 - 15.31 - 5.45) \\ -(20.46) + 1.85\,(15.31 + 14.37 - 3.77 - 5.45) \\ -(-23.97) + 1.85\,(+82.79 - 14.37 - 9.60 - 82.79) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \\ 5 \\ 5 \\ 5 \\ 3 \\ 165.93 \\ -6.29 \\ 17.39 \\ -20.37 \end{bmatrix}$$

The system of linear equations ($\mathbf{J_{NL}}\ \mathbf{Q}^{(1)} = \mathbf{F_{NL}}$, Eq. 5-24) are then solved for the unknown $\mathbf{Q}^{(1)}$:

$$\mathbf{Q}^{(1)\,T} = [20.94, 9.42, 9.42, 11.53, 5.88, 6.17, 3.24, 0.36, 1.17, 0.46, 0.600, 7.06]$$

Updating $\mathbf{J_{NL}}$ and the RHS values, the method converges to the solution in 3 iterations as shown in Table E5-6.

Table E5-6: Pipe flows for 4 iterations of the modified linear theory method for the pipe equations.

	Pipe										
m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	***P***
0	20	9	11	6	5.5	3.5	0.5	0.5	1	1	8
1	20.94	9.42	11.53	5.88	6.17	3.24	0.36	1.17	0.46	0.60	7.06
2	21.21	9.79	11.43	6.04	6.07	3.39	0.36	1.07	0.25	0.54	6.79
3	21.27	9.84	11.43	6.03	6.07	3.39	0.36	1.07	0.19	0.54	6.73
4	21.27	9.84	11.43	6.03	6.07	3.39	0.36	1.07	0.19	0.54	6.73

5.3.3 Solution of the Node Equations (Simultaneous Node Adjustment Algorithm)

The pipe head loss equation for pipe l that connects nodes i and j:

$$h_{L,l} = H_j - H_i = K_l \lfloor Q_l \rfloor^n \tag{5-27}$$

can be transformed to nodal head equation as:

$$Q_l = \left(\frac{\lfloor H_j - H_i \rfloor}{K_l} \right)^{1/n} \tag{5-28}$$

Shamir and Howard (1968) solved used this transformation to form *nnode* node equations for the nodal heads using the Newton-Raphson method. Substituting Eq. 5-28 in Eq. 5-24 for a general node i gives:

$$F_{N,i} : \sum_{l \in J_{in}, J_{out}} \left(\frac{\lfloor H_j - H_i \rfloor}{K_l} \right)^{1/n} = q_i \tag{5-29}$$

where the summation is over the pipes entering or leaving the node. In the head difference, the node for which the mass balance is written is always the second term. The sign notation, $\lfloor H_j - H_i \rfloor$, provides the flow direction. If H_i is greater than H_j, flow is from node i towards j (an outflow) and the sign is negative. When H_j exceeds H_i, the sign is positive and flow is supplied to node i from node j (an inflow).

For node i, application of the Newton-Raphson method yields:

$$\sum_{i \in ncnode(i(} \left(\left. \frac{\partial F_{N,i}}{\partial (\Delta H_i)} \right|_{\Delta H = 0} \Delta H_i \right) = - F(H^{(m-1)}) \qquad (5\text{-}30)$$

where $H^{(m-1)}$ are the nodal heads for the previous and present iterations and $ncnode(i)$ is the set of nodes that are connected by pipes to node i and node i (e.g., nodes 2, 4, 6, and 7 for node 7). In matrix form for all equations and nodes:

$$\mathbf{J_N}\, \Delta\mathbf{H} = -\, \mathbf{F_N} \qquad (5\text{-}31)$$

where $\mathbf{J_N}$ is the Jacobian matrix of the node equations with respect to the changes in nodal heads and $\mathbf{F_N}$ is the residuals of the node equations. Both $\mathbf{J_N}$ and $\mathbf{F_N}$ are evaluated at $\Delta\mathbf{H}$ equal to zero or the present iteration's head estimates. It should be noted that the square Jacobian matrix (*nnode* x *nnode*) is symmetric and positive definite.

For pipe l that connects nodes i and j, the Jacobian terms gradients for the conservation of mass at node i are:

$$\left. \frac{\partial F_{N,i}}{\partial (\Delta H_j)} \right|_{H^{m-1}} = - \frac{1}{n\, K_l} \left(\frac{\lfloor H_j - H_i \rfloor}{K_l} \right)^{\left(\frac{1}{n}-1\right)} = - \left| \frac{Q_l}{n\,(H_j - H_i)} \right| \qquad (5\text{-}32a)$$

$$\left. \frac{\partial F_{N,i}}{\partial (\Delta H_i)} \right|_{H^{m-1}} = \sum_{l \in ncp(i)} \frac{1}{n\, K_l} \left(\frac{\lfloor H_j - H_i \rfloor}{K_l} \right)^{\left(\frac{1}{n}-1\right)} = \sum_{l \in ncp(i)} \left| \frac{Q_l}{n\,(H_j - H_i)} \right| \qquad (5\text{-}32b)$$

Regardless of the flow direction, the gradient sign for the flow balance at node i are all positive for H_i terms while the terms for the connecting nodes j are all negative.

After Eq. 5-31 are solved for $\Delta\mathbf{H}$, the heads are updated by subtracting the nodal corrections or:

$$H_i^{(m)} = H_i^{(m-1)} - \Delta H_i \qquad (5\text{-}33)$$

The process is completed iteratively until the changes in nodal heads for all nodes are less than a tolerance or a desired number of iterations are completed.

The overall process is:

1) Initialize $m = 0$ and define starting set of nodal heads, $H^{(0)}$.
2) Set $m = m + 1$
3) Compute nodal balances using Eq. 5-29 and gradients using Eq. 5-32
4) Solve the system of equations 5-31 for $\Delta\mathbf{H}$
5) Update nodal heads using Eq. 5-33
6) Check stopping criteria. If satisfied, stop. If not satisfied, go to step 2.

Example 5.7

Problem: Determine the nodal heads at all nodes in the three loop network in Example 5.4 using the modified linear theory method and the Hazen-Williams equation. Assume an initial head vector,

$$\mathbf{H}^{(0),T} = [198, 193, 195, 175, 188, 190, 184]$$

Compute the flow rate in pipe 4.

Solution: Given the initial head distribution (step 1), we can now update the values at m=1 (step 2).

Step 3: As listed in Example 5.6, the nodal mass balance equations are:

$$\begin{aligned}
Node\ 1: Q_1 - Q_2 - Q_3 = q_1 = 0 \\
Node\ 2: Q_2 - Q_4 + Q_9 = q_2 = 4 \\
Node\ 3: -Q_9 - Q_{10} + Q_P = q_3 = 6 \\
Node\ 4: Q_6 + Q_8 + Q_{10} = q_4 = 5 \\
Node\ 5: Q_5 - Q_8 = q_5 = 5 \\
Node\ 6: Q_3 - Q_5 - Q_7 = q_6 = 5 \\
Node\ 7: Q_4 + Q_7 - Q_6 = q_7 = 3
\end{aligned}$$

For the node equation solution, these seven equations are written in terms of the seven nodal heads as:

$$F_{N,1} = \left(\frac{\lfloor H_{res.1} - H_1 \rfloor}{K_1}\right)^{0.54} + \left(\frac{\lfloor H_2 - H_1 \rfloor}{K_2}\right)^{0.54} + \left(\frac{\lfloor H_6 - H_1 \rfloor}{K_3}\right)^{0.54} - q_1 = 0$$

$$F_{N,2} = \left(\frac{\lfloor H_1 - H_2 \rfloor}{K_2}\right)^{0.54} + \left(\frac{\lfloor H_7 - H_2 \rfloor}{K_4}\right)^{0.54} + \left(\frac{\lfloor H_3 - H_2 \rfloor}{K_9}\right)^{0.54} - q_2 = 0$$

$$F_{N,3} = \left(\frac{\lfloor H_2 - H_3 \rfloor}{K_9} \right)^{0.54} + \left(\frac{\lfloor H_4 - H_3 \rfloor}{K_{10}} \right)^{0.54} + \left(\frac{\lfloor H_{res.2} - H_3 - 240 \rfloor}{-0.9376} \right)^{0.5} - q_3 = 0$$

$$F_{N,4} = \left(\frac{\lfloor H_7 - H_4 \rfloor}{K_6} \right)^{0.54} + \left(\frac{\lfloor H_5 - H_4 \rfloor}{K_8} \right)^{0.54} + \left(\frac{\lfloor H_3 - H_4 \rfloor}{K_{10}} \right)^{0.54} - q_4 = 0$$

$$F_{N,5} = \left(\frac{\lfloor H_6 - H_5 \rfloor}{K_5} \right)^{0.54} + \left(\frac{\lfloor H_4 - H_5 \rfloor}{K_8} \right)^{0.54} - q_5 = 0$$

$$F_{N,6} = \left(\frac{\lfloor H_1 - H_6 \rfloor}{K_3} \right)^{0.54} + \left(\frac{\lfloor H_5 - H_6 \rfloor}{K_5} \right)^{0.54} + \left(\frac{\lfloor H_7 - H_6 \rfloor}{K_7} \right)^{0.54} - q_6 = 0$$

$$F_{N,7} = \left(\frac{\lfloor H_2 - H_7 \rfloor}{K_4} \right)^{0.54} + \left(\frac{\lfloor H_6 - H_7 \rfloor}{K_7} \right)^{0.54} + \left(\frac{\lfloor H_4 - H_7 \rfloor}{K_6} \right)^{0.54} - q_7 = 0$$

Since this form is nonlinear, all *nnode* equations are linearized about the *nnode* corrections.

To set up the linear equations, the pipe flows are computed using the previously computed *K*'s (Table E5-4a) and the initial nodal heads and are listed in Table E5-7a. For pipe 1 and node 1:

$$Q_1^{(0)} = \left(\frac{\lfloor H_{res.1} - H_1 \rfloor}{K_1} \right)^{0.54} = \left(\frac{\lfloor 200 - 198 \rfloor}{0.00584} \right)^{0.54} = 23.368\ cfs$$

The gradient of the flow rate in pipe 1 with respect to a change in head at node one is given by Eq. 5-32b:

$$\left. \frac{\partial F_1}{\partial \Delta H_1} \right|_{H_1^{(0)}} = \frac{(H_{res.1} - H_1)^{-0.46}}{n\, K_1^{0.54}} = \frac{(H_{res.1} - H_1)^{0.54}}{n\,(H_{res.1} - H_1)\, K_1^{0.54}}$$

$$= \left| \frac{Q_1}{n\,(H_{res.1} - H_1)} \right| = \frac{23.368}{1.85\ (2)} = 6.316$$

The gradient term for each pipe is listed in the last row of Table E5-7a.

Table E5-7a: Pipe flows computed with the defined initial nodal heads.

Pipe	1	2	3	4	5
K	0.00584	0.0645	0.0645	0.349	0.233
$(H_i - H_j)$	2	5	8	9	2
$Q^{(0)}$	23.368	10.474	13.50	5.785	3.20
$Q/n\,(H_i - H_j)$	6.316	1.131	0.911	0.347	0.863

Table E5-7a (cont.): Pipe flows computed with the defined initial nodal heads.

Pipe	6	7	8	9	10
K	1.416	55.20	13.60	82.79	82.79
$(H_i - H_j)$	9	6	13	2	20
$Q^{(0)}$	2.715	0.302	0.976	0.134	0.464
$Q/n\,(H_i - H_j)$	0.163	0.027	0.041	0.036	0.013

For the first iteration, the *nnode* x *nnode* coefficient matrix, $\mathbf{J_N}$, is:

$$\mathbf{J_N} = \begin{bmatrix} 8.352 & -1.131 & 0 & 0 & 0 & -0.911 & 0 \\ -1.131 & 1.514 & -0.036 & 0 & 0 & 0 & -0.347 \\ 0 & -0.036 & 0.126 & -0.013 & 0 & 0 & 0 \\ 0 & 0 & -0.013 & 0.216 & -0.041 & 0 & -0.163 \\ 0 & 0 & 0 & -0.041 & 0.904 & -0.863 & 0 \\ -0.911 & 0 & 0 & 0 & -0.863 & 1.802 & -0.027 \\ 0 & -0.347 & 0 & -0.163 & 0 & -0.027 & 0.537 \end{bmatrix}$$

The diagonal terms are the sum of the gradients for that node. For node 2,

$$\frac{\partial F_2}{\partial(\Delta H_2)} = \left|\frac{Q_2^{(0)}}{n\,(H_1^{(0)} - H_2^{(0)})}\right| + \left|\frac{Q_9^{(0)}}{n\,(H_3^{(0)} - H_2^{(0)})}\right| + \left|\frac{Q_4^{(0)}}{n\,(H_2^{(0)} - H_7^{(0)})}\right|$$
$$= 1.131 + 0.347 + 0.036 = 1.514$$

where the individual terms were taken from Table E5-7a. The off-diagonal terms are equal to the gradients of the node equation with respect to the adjacent node and are always negative.

For example, the change in node 2's mass balance due to a change in head at node 3 is:

$$\frac{\partial F_2}{\partial(\Delta H_3)} = -\left|\frac{Q_9^{(0)}}{n\,(H_3^{(0)} - H_2^{(0)})}\right| = -0.036 = \frac{\partial F_3}{\partial(\Delta H_2)}$$

This term is placed in row two for mass balance at node 2-column 3 for connecting node 3. A change in the head at node 2 causes an equal change in the mass balance at node 3. So the term also appears in row (node equation) three – column (connecting node) two.

The right hand side of the system of equations is $\mathbf{F_N}$, the residuals of the nodal balance equations. They are computed by substituting the summed head values in the mass balance equations listed above. For example for node 6 at iteration 0 the residual is:

$$F_{N,6} = \left(\frac{\lfloor H_1 - H_6 \rfloor}{K_3}\right)^{0.54} + \left(\frac{\lfloor H_5 - H_6 \rfloor}{K_5}\right)^{0.54} + \left(\frac{\lfloor H_6 - H_7 \rfloor}{K_7}\right)^{0.54} - q_6 =$$

$$= \left(\frac{(198-190)}{0.0645}\right)^{0.54} - \left(\frac{|188-190|}{0.233}\right)^{0.54} - \left(\frac{|184-190|}{55.2}\right)^{0.54} - 5 = 5.002$$

Their values are shown in first row of Table E5-7b. The signs of these terms are then changed in the solution of Eq. 5-31. The set of equations 5-31 is then:

$$\mathbf{J_N}\Delta\mathbf{H} = \begin{bmatrix} 8.352 & -1.131 & 0 & 0 & 0 & -0.911 & 0 \\ -1.131 & 1.514 & -0.036 & 0 & 0 & 0 & -0.347 \\ 0 & -0.036 & 0.126 & -0.013 & 0 & 0 & 0 \\ 0 & 0 & -0.013 & 0.216 & -0.041 & 0 & -0.163 \\ 0 & 0 & 0 & -0.041 & 0.904 & -0.863 & 0 \\ -0.911 & 0 & 0 & 0 & -0.863 & 1.802 & -0.027 \\ 0 & -0.347 & 0 & -0.163 & 0 & -0.027 & 0.537 \end{bmatrix} \begin{bmatrix} \Delta H_1 \\ \Delta H_2 \\ \Delta H_3 \\ \Delta H_4 \\ \Delta H_5 \\ \Delta H_6 \\ \Delta H_7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.606 \\ -0.822 \\ -0.330 \\ 0.845 \\ 2.779 \\ -5.002 \\ -0.372 \end{bmatrix} = -\mathbf{F_N}$$

Step 4: The first iteration solution for $\Delta\mathbf{H}^{\mathrm{T}}$ = [-0.333, -0.921, -2.520, 3.684, 0.782, -2.575, -0.301].

Step 5: The updated nodal heads are listed in Table E5-7c where $H^{(1)} = H^{(0)} - \Delta H$. For node 1:

$$H_1^{(1)} = H_1^{(0)} - \Delta H_1 = 198.0 - (-0.333) = 198.333$$

Step 6: A large change in nodal heads was applied so return to step 2 (m=2). The nodal mass balance equations are evaluated again and mass balance has not been achieved (Table E5-7b). The nodal heads are updated two more times and the values converge as seen in Table E5-7c.

After the flows have converged after iteration 3, the flow rate in pipe 4 can be computed by:

$$Q_4^{(final)} = \left(\frac{\lfloor H_2 - H_7 \rfloor}{K_4}\right)^{0.54} = \left(\frac{193.863 - 184.145}{0.349}\right)^{0.54} = 6.028\,cfs$$

Table E5-7b: Residuals of the nodal balance equations, F_N, computed using the nodal heads at the beginning of iteration. To solve the set of equations these residuals are multiplied by –1 in Equation 5-31.

	Node						
Iteration	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	-0.606	0.822	0.330	-0.845	-2.779	5.002	0.372
1	0.085	-0.024	0.010	-0.0.65	-0.646	0.503	0.047
2	-0.001	0	0	-0.001	-0.028	0.027	0.001
3	0	0	0	0	0	0	0

Table E5-7c: Nodal heads for the three iterations.

	Node						
Iteration	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	198	193	195	175	188	190	184
1	198.333	193.921	197.52	171.316	187.218	192.575	184.301
2	198.320	193.864	197.523	170.603	185.948	192.445	184.154
3	198.320	193.863	197.522	170.584	185.860	192.446	184.145

5.3.4 Solution of the Pipe Equations

In the loop equation formulation, head losses were balanced around a series of pipes between points with a known difference in energy. Hamam and Brameller (1972) for gas networks and Todini and Pilati (1987) for water networks wrote conservation of energy for each pipe (Eq. 5-14) resulting in a set of *npipe* equations with the *npipe* pipe flows and the *nnode* nodal heads as unknown. They coupled these equations with the node equations written in terms of the pipe flows (Eq. 5-11) to form a set of *npipe* plus *nnode* equations for an equal number of unknowns. The method is also known as the hybrid or gradient approach. For additional background on the method, see Osiadacz (1991).

For the four-pipe network shown in Figure 5-3, the pipe equations include one equation for each node and each pipe. With the assumed set of pipe flow directions, the node equations are the conservation of mass relationships or:

$$F_{Q2}:\ +Q_1 - Q_2 - Q_3 - q_2 = 0 \quad \{node\ 2\}$$
$$F_{Q3}:\ +Q_2 - Q_4 - q_3 = 0 \quad \{node\ 3\}$$
$$F_{Q4}:\ +Q_3 + Q_4 - q_4 = 0 \quad \{node\ 4\}$$

The pipe equations are written for each pipe with the nodal head and pipe flow on the left-hand side of the equation. The friction loss equation is given a positive sign, so the upstream source node head has a negative sign and the downstream head takes a positive sign. The equations for pipes 1 to 4 in the four-pipe network are:

$$F_{P,1}:\ K_1 \lfloor Q_1 \rfloor^n + H_2 - H_1 = 0 \quad \{pipe\ 1\}$$
$$F_{P,2}:\ K_2 \lfloor Q_2 \rfloor^n - H_2 + H_3 = 0 \quad \{pipe\ 2\}$$
$$F_{P,3}:\ K_3 \lfloor Q_3 \rfloor^n - H_2 + H_4 = 0 \quad \{pipe\ 3\}$$
$$F_{P,4}:\ K_4 \lfloor Q_4 \rfloor^n - H_3 + H_4 = 0 \quad \{pipe\ 4\}$$

where H_1 and all K's are known.

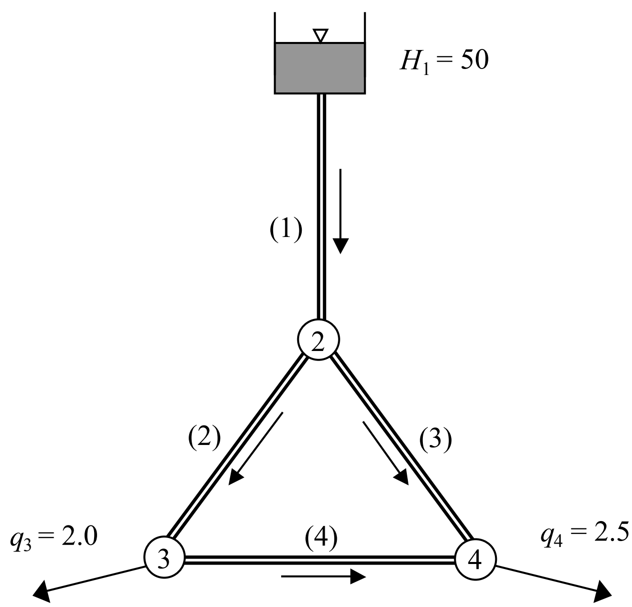


Figure 5-3: Four pipe example network for pipe equations formulation.

The notation, $\lfloor Q_1 \rfloor^n$, represents that the absolute value of the pipe flow 1 is raised to the power *n* and the sign of the pipe flow is applied to the head loss equation term. The flow's sign also is applied in the node equations. Thus, a negative flow is acceptable and defines a flow that is in the opposite direction from the initial assumption. In our example, conservation of mass and energy comprise seven equations written with respect to four pipe flows and three nodal heads. Applying the Newton-Raphson method (but solving for the changes in flow and head) for pipe *l* that connects nodes *i* and *j* gives:

$$\sum_{l \in J_{in}, J_{out}} \left(\left. \frac{\partial F_Q}{\partial Q_l} \right|_{\Delta Q=0} \Delta Q_l^{(m)} \right) = -F_Q(Q^{(m-1)}) \qquad (5\text{-}34)$$

$$\left. \frac{\partial F_P}{\partial H_i} \right|_{\Delta H=0} \Delta H_i^{(m)} + \left. \frac{\partial F_P}{\partial H_j} \right|_{\Delta H=0} \Delta H_j^{(m)} + \left. \frac{\partial F_P}{\partial Q_l} \right|_{\Delta Q=0} \Delta Q_l^{(m)} = -F_P(Q^{(m-1)}, H^{(m-1)}) \qquad (5\text{-}35)$$

The derivatives of the mass balance equations (F_Q) are 1 (outflow pipe), -1 (inflow pipe) or 0 (not connected to node). The right-hand side is calculated by substituting in the present estimates of the flow rates and defined as *dq*. For node 2, Eq. 5-34 becomes:

$$\left. \frac{\partial F_Q}{\partial Q_1} \right|_{\Delta Q=0} \Delta Q_1^{(m)} + \left. \frac{\partial F_Q}{\partial Q_2} \right|_{\Delta Q=0} \Delta Q_2^{(m)} + \left. \frac{\partial F_Q}{\partial Q_3} \right|_{\Delta Q=0} \Delta Q_3^{(m)} = -F_{Q1}(Q^{(m-1)}) =$$

$$(+1)\,\Delta Q_1 + (-1)\Delta Q_2 + (-1)\Delta Q_3 = -(+Q_1^{(m-1)} - Q_2^{(m-1)} - Q_3^{(m-1)} - q_2) = -\,dq_2 \qquad \{F_{Q,2}\text{: node } 2\} \qquad (5\text{-}36a)$$

Similarly for nodes 3 and 4:

$$+\Delta Q_2 - \Delta Q_4 = -(+Q_2^{(m-1)} - Q_4^{(m-1)} - q_3) = -\,dq_3 \qquad \{F_{Q,3}\text{: node } 3\} \qquad (5\text{-}36b)$$

$$+\Delta Q_3 + \Delta Q_4 = -(+Q_3^{(m-1)} + Q_4^{(m-1)} - q_4) = -\,dq_4 \qquad \{F_{Q,4}\text{: node } 4\} \qquad (5\text{-}36c)$$

For the energy balances, the derivatives with respect to the nodal heads are 1 (sink node), -1 (source node), and 0 (not connected to pipe). The derivatives with respect to pipe flow are $n\,K\left|Q\right|^{n-1}$. The right-hand side is computed using the present iterations flows and nodal heads and defined as *dE*. For pipe 1, Eq. 5-35 is:

$$n\,K_1\left|Q_1\right|^{n-1}\Delta Q_1 + (+1)\,\Delta H_2 = -(H_2 + K_1 Q_1^{\,n} - H_1) = -dE_1 \quad \{F_{P1}\text{: } pipe\ 1\} \quad (5\text{-}37a)$$

The gradient for the head at node one is zero since it is a fixed head reservoir. Node 2 is the downstream (sink) node so its gradient is positive and the pipe flow term is evaluated at the present $Q^{(m-1)}$. For clarity the m-1 iteration counter is not included in all pipe equations.

For pipes 2 to 4, the gradient equations are:

$$n\,K_2\left|Q_2\right|^{n-1}\Delta Q_2 - \Delta H_2 + \Delta H_3 = -(-H_2 + H_3 + K_2 Q_2^{\,n}) = -\,dE_2 \quad \{F_{P2}\text{: } pipe\ 2\} \quad (5\text{-}37b)$$

$$n\,K_3\left|Q_3\right|^{n-1}\Delta Q_3 - \Delta H_2 + \Delta H_4 = -(-H_2 + H_4 + K_3 Q_3^{\,n}) = -\,dE_3 \quad \{F_{P3}\text{: } pipe\ 3\} \quad (5\text{-}37c)$$

$$n\,K_4\left|Q_4\right|^{n-1}\Delta Q_4 - \Delta H_3 + \Delta H_4 = -(+H_3 - H_4 + K_4 Q_4^{\,n}) = -\,dE_4 \quad \{F_{P4}\text{: } pipe\ 4\} \quad (5\text{-}37d)$$

These seven equations (Eq. 5-36 a-c and 5-37 a-d) can be solved for the changes in nodal head and flow. The new iterations values are then computed by:

$$H^{(m)} = H^{(m-1)} + \Delta H^{(m)} \quad (5\text{-}38)$$

$$Q^{(m)} = Q^{(m-1)} + \Delta Q^{(m)} \quad (5\text{-}39)$$

Todini and Pilati (1987) generalized this formulation in matrix form. Conservation of energy (in the pipes) and mass (at the nodes) equations (Eq. 5-16 and 5-24) can be written in matrix form as:

$$\mathbf{F_P(Q,H) = A}_{11}\,\mathbf{Q}^{(m)} + \mathbf{A}_{12}\,\mathbf{H}^{(m)} = \mathbf{0} \quad (5\text{-}40)$$

$$\mathbf{F_Q(Q,H) = A}_{21}\,\mathbf{Q}^{(m)} - \mathbf{q} = \mathbf{0} \quad (5\text{-}41)$$

respectively.

Looking back at the node and pipe equations written at the beginning of this section, you will note that the coefficients on the flows entering and leaving node 2 are 1, -1, -1, and 0 for pipe flows 1-4, respectively. These values are identical to the coefficients in the column related to nodal head H_2 in the pipe equations. These coefficients comprise the matrices, $\mathbf{A}_{12}$ and $\mathbf{A}_{21}$. $\mathbf{A}_{21}$ is the connectivity, also known as the topological, matrix and $\mathbf{A}_{21} = \mathbf{A}_{12}^{T}$. The terms in the $\mathbf{A}_{12}$ matrix identify the network connections and take on values of 1, -1 and 0. Each column corresponds to a pipe and values of -1 are assigned to the upstream node for the pipe, 1 to outlet node of the pipe and 0 if the pipe is not connected to the node. For the four pipe network,

$$\mathbf{A}_{21} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{A}_{12} = \mathbf{A}_{21}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

$\mathbf{A}_{11}$ is defined as:

$$\mathbf{A}_{11} = \begin{bmatrix} K_1 \left|Q_1\right|^{n-1} & & 0 \\ & \ddots & \\ 0 & & K_{npipe} \left|Q_{npipe}\right|^{n-1} \end{bmatrix} \qquad (5\text{-}42)$$

Note that the rows in $\mathbf{A}_{21}$ correspond to nodes 2 – 4.

Applying the Newton-type solution scheme to the system of equations gives:

$$\mathbf{dF_P(Q,H)} = n\,\mathbf{A}_{11}\,\Delta\mathbf{Q}^{(m)} + \mathbf{A}_{12}\,\Delta\mathbf{H}^{(m)} = -\,\mathbf{dE} \qquad (5\text{-}43)$$

$$\mathbf{dF_Q(Q,H)} = \mathbf{A}_{21}\,\Delta\mathbf{Q}^{(m)} = -\,\mathbf{dq} \qquad (5\text{-}44)$$

The resulting equations for the four-pipe network are given in Eqs. 5-37a-d and 5-36a-c. The right-hand side terms are shown for the four-pipe network as the residuals in the mass and energy balance equations at iteration m-1. This set of equations is solved for $\Delta\mathbf{Q}$ and $\Delta\mathbf{H}$ and $\mathbf{H}^{(m)}$ and $\mathbf{Q}^{(m)}$ are updated by:

$$\mathbf{H}^{(m)} = \mathbf{H}^{(m-1)} + \Delta\mathbf{H}^{(m)} \qquad (5\text{-}45)$$

$$\mathbf{Q}^{(m)} = \mathbf{Q}^{(m-1)} + \Delta\mathbf{Q}^{(m)} \qquad (5\text{-}46)$$

As in the earlier methods, the absolute or relative changes in flow or head or the number of iterations can be used as stopping criteria.

The overall procedure for solving the pipe equations is:

1) Initialize $m = 0$ and define starting set of nodal heads, $\mathbf{H}^{(0)}$, and pipe flows, $\mathbf{Q}^{(0)}$
2) Form matrix $\mathbf{A}_{12}$
3) Set $m = m + 1$
4) Form matrix $\mathbf{nA}_{11}$ using Eq. 5-42.
5) Compute nodal balance error ($\mathbf{dF_Q = -dq}$) using Eq. 5-16 and pipe balance error ($\mathbf{dF_H = -dE}$) with Eq. 5-24
6) Solve system of equations (Eq. 5-43 and 5-44) for $\Delta\mathbf{H}$ and $\Delta\mathbf{Q}$
7) Update nodal heads and pipe flows using Eqs. 5-45 and 5-46.
8) Check stopping criteria. If satisfied, stop. If not satisfied, go to step 3.

Assume that all pipes are identical in the four-pipe network and the Hazen-Williams equation is applied with $K = 0.935$ and $n = 1.852$. Also assume that the flows in the four pipes are [4.5, 2, 2, 0.5] and the nodal heads are [40, 35, 30]. The head in the reservoir, H_1, is 50. The $\mathbf{A}_{21}$ is given above and $\mathbf{A}_{11}$ diagonal matrix is computed by $KQ^{n\text{-}1} = KQ^{0.852}$. For pipe 1, $K|Q|^{0.852} = 0.935\,(|4.5|)^{0.852} = 3.37$ and the term in the solution matrix is $1.852\,K|Q|^{0.852} = 6.24$. The overall left hand side matrix is:

$$\left[\begin{array}{c|c} n\,\mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|ccc} 6.24 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3.13 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3.13 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0.96 & 0 & -1 & 1 \\ \hline 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

where the upper left portion is $nK|Q|^{n\text{-}1}$, the upper right is $\mathbf{A}_{12}$, the lower left is $\mathbf{A}_{21}$, and the lower right are zeros corresponding to the node equation coefficients on the nodal heads.

The right hand side of the equations is the errors in the equations. For pipe 1, dE_1 is computed from Eq 5-37a:

$$-(K_1\,Q_1^{\,n} + H_2 - H_1) = -\,dE_1 = -(0.935\,(4.5)^{1.852} + 40 - 50) = -\,5.16$$

For node 2 (Eq. 5-36a):

$$-(+Q_1 - Q_2 - Q_3 - q_2) = -dq_2 = -(4.5 - 2 - 2 - 0) = -\,0.5$$

After computing values for each equation, the transpose of the right hand side vector is:

$$\left[-\mathbf{dE} \mid -\mathbf{dq}\right]^T = \left[-5.16 \quad 1.62 \quad 6.62 \quad 4.74 \;\vert\; -0.5 \quad 0.5 \quad 0\right]^T$$

and the full matrix equation is:

$$\left[\begin{array}{c|c} n\,\mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & 0 \end{array}\right]\left[\begin{array}{c} \Delta\mathbf{Q} \\ \hline \Delta\mathbf{H} \end{array}\right] = \left[\begin{array}{c} -\mathbf{dE} \\ \hline -\mathbf{dq} \end{array}\right]$$

$$= \left[\begin{array}{cccc|ccc} 6.24 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3.13 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3.13 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0.96 & 0 & -1 & 1 \\ \hline 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} \Delta Q_1 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \\ \hline \Delta H_2 \\ \Delta H_3 \\ \Delta H_4 \end{array}\right] = \left[\begin{array}{c} -5.16 \\ 1.62 \\ 6.62 \\ 4.74 \\ \hline -0.5 \\ 0.5 \\ 0 \end{array}\right]$$

Solving for $\Delta\mathbf{Q}$ and $\Delta\mathbf{H}$ gives:

$$\left[\Delta\mathbf{Q} \mid \Delta\mathbf{H}\right]^T = \left[0 \quad 0.247 \quad 0.253 \quad -0.253 \;\vert\; -5.16 \quad -4.30 \quad 0.68\right]^T$$

Substituting this vector in Eq. 5-45 and 5-46 gives the next iteration's pipe flow and nodal heads or:

$$\left[\mathbf{Q} \mid \mathbf{H}\right]^T = \left[4.5 \quad 2.247 \quad 2.253 \quad 0.247 \;\vert\; 34.84 \quad 30.70 \quad 30.68\right]^T$$

Repeating the process for another iteration gives the final solution of:

$$\left[\mathbf{Q} \mid \mathbf{H}\right]^T = \left[4.5 \quad 2.24 \quad 2.26 \quad 0.24 \;\vert\; 34.85 \quad 30.69 \quad 30.62\right]^T$$

Example 5.8

Problem: For the Example 5.4 network, determine the flow rates in the pipes and the nodal heads at all nodes using the gradient method and the Hazen-Williams equation. Assume an initial head vector, $\mathbf{H}^{(0)\,T}$ = [198, 193, 195, 175, 188, 190, 184] and an initial pipe flow vector, $\mathbf{Q}^{(0)\,T}$ = [20, 9, 11, 6, 5.5, 3.5, 0.5, 0.5, 1, 1, 8].

Solution: Given the initial head and flow distribution (step 1), we can now update the values at m=1 (step 2). The set of equations for pipe equations are comprised of the head loss relationship for each pipe in terms of the nodal heads and pipe flows (conservation of energy):

$$Pipe\ 1:\ h_{L,1} = H_{res\,1} - H_1 = K_1 \lfloor Q_1 \rfloor^n \Rightarrow K_1 \lfloor Q_1 \rfloor^n - H_{res\,1} + H_1 = 0$$
$$Pipe\ 2:\ h_{L,2} = H_1 - H_2 = K_2 \lfloor Q_2 \rfloor^n \Rightarrow K_2 \lfloor Q_2 \rfloor^n - H_1 + H_2 = 0$$
$$Pipe\ 3:\ h_{L,3} = H_1 - H_6 = K_3 \lfloor Q_3 \rfloor^n \Rightarrow K_3 \lfloor Q_3 \rfloor^n - H_1 + H_6 = 0$$
$$Pipe\ 4:\ h_{L,4} = H_2 - H_7 = K_4 \lfloor Q_4 \rfloor^n \Rightarrow K_4 \lfloor Q_4 \rfloor^n - H_2 + H_7 = 0$$
$$Pipe\ 5:\ h_{L,5} = H_6 - H_5 = K_5 \lfloor Q_5 \rfloor^n \Rightarrow K_5 \lfloor Q_5 \rfloor^n - H_6 + H_5 = 0$$
$$Pipe\ 6:\ h_{L,6} = H_7 - H_4 = K_6 \lfloor Q_6 \rfloor^n \Rightarrow K_6 \lfloor Q_6 \rfloor^n - H_7 + H_4 = 0$$
$$Pipe\ 7:\ h_{L,7} = H_6 - H_7 = K_7 \lfloor Q_7 \rfloor^n \Rightarrow K_7 \lfloor Q_7 \rfloor^n - H_6 + H_7 = 0$$
$$Pipe\ 8:\ h_{L,8} = H_5 - H_4 = K_8 \lfloor Q_8 \rfloor^n \Rightarrow K_8 \lfloor Q_8 \rfloor^n - H_5 + H_4 = 0$$
$$Pipe\ 9:\ h_{L,9} = H_3 - H_2 = K_9 \lfloor Q_9 \rfloor^n \Rightarrow K_9 \lfloor Q_9 \rfloor^n - H_3 + H_2 = 0$$
$$Pipe\ 10:\ h_{L,10} = H_3 - H_4 = K_{10} \lfloor Q_{10} \rfloor^n \Rightarrow K_{10} \lfloor Q_{10} \rfloor^n - H_3 + H_4 = 0$$
$$Pump:\ \begin{array}{l} h_p = H_{res\,2} - H_3 = -(240 - 0.9376\, Q_p^2) \\ \Rightarrow -(240 - 0.9376\, Q_p^2) - H_{res\,2} + H_3 = 0 \end{array}$$

and the conservation of mass at each node in terms of the pipe flows:

$$Node\,1: Q_1 - Q_2 - Q_3 - q_1 = 0$$
$$Node\,2: Q_2 - Q_4 + Q_9 - q_2 = 0$$
$$Node\,3: -Q_9 - Q_{10} + Q_P - q_3 = 0$$
$$Node\,4: Q_6 + Q_8 + Q_{10} - q_4 = 0$$
$$Node\,5: Q_5 - Q_8 - q_5 = 0$$
$$Node\,6: Q_3 - Q_5 - Q_7 - q_6 = 0$$
$$Node\,7: Q_4 - Q_6 + Q_7 - q_7 = 0$$

From the node equations, $\mathbf{A}_{21}$ is:

$$\mathbf{A}_{21} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of $\mathbf{A}_{21}$ represent nodes and the columns of the pipes. For example, row 1 corresponds to node 1. The values in the matrix equal the coefficients on the pipe flows in the node equation. Row 1 is then 1 for pipe 1, -1 for pipes 2 and 3, and 0 for all others. Column 11 corresponds to the pump flow and only has a non-zero for node (row) 3. Examining the pipe equations and the coefficients on the nodal heads we can confirm that $\mathbf{A}_{21} = \mathbf{A}_{12}^T$.

Step 3: m=1

Step 4: At iteration 1, the initial flow values are used to compute matrix $\mathbf{A}_{11}$. For pipe 1, the diagonal term is:

$$nK_1 \left|Q_1^{(0)}\right|^{n-1} = 1.85\,(0.00584\left|20\right|^{0.85}) = 0.138$$

The full $n\mathbf{A}_{11}$ matrix is:

$$n\mathbf{A}_{11} = \begin{bmatrix} 0.138 & & & & & & & & & & \\ & 0.774 & & & & & & & \cdots & 0 & \\ & & 0.917 & & & & & & & & \\ & & & 2.959 & & & & & & \vdots & \\ & & & & 1.832 & & & & & & \\ & & & & & 7.596 & & & & & \\ & & & & & & 56.651 & & & & \\ & \vdots & & & & & & 13.956 & & & \\ & & & & & & & & 153.17 & & \\ & 0 & & \cdots & & & & & & 153.17 & \\ & & & & & & & & & & 15.002 \end{bmatrix}$$

Step 5: It is unlikely that the initial guess will be the correct flow and nodal head values. Therefore, the conservation laws (Eq. 5-16 and 5-24 and written above for the example network) will not be satisfied. To determine the residuals (**dE** and **dq**), the assumed values $\mathbf{H}^{(0)}$ and $\mathbf{Q}^{(0)}$ are substituted. Since we developed the flows so that they balance at node, **dq** equals zero but **dE** does not. For pipe 1:

$$dE_1 = K_1 \lfloor Q_1 \rfloor^n - H_{res.1} + H_1 = 0.00584\,(20)^{1.85} - 200 + 198 = -\,0.509$$

The full vector is:

$$\mathbf{dE}^T = [-0.509, -1.240, -2.549, -0.576, 3.447, 5.371, 9.311, -9.228, 80.794, 62.794, 15.006]^T$$

and

$$\mathbf{dq}^{T} = [0, 0, 0, 0, 0, 0, 0]^{T}$$

Step 6: The vector computed in step 5 is the right hand side of the system of equations.

$$\left[\begin{array}{c|c} n\,\mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & 0 \end{array}\right]\left[\begin{array}{c} \Delta\mathbf{Q} \\ \hline \Delta\mathbf{H} \end{array}\right] = \left[\begin{array}{c} -\mathbf{dE} \\ \hline -\mathbf{dq} \end{array}\right]$$

The unknowns are the vectors, $\Delta\mathbf{H}$ and $\Delta\mathbf{Q}$. The coefficient matrix is (with truncated $n\mathbf{A}_{11}$ terms):

$$\left[\begin{array}{c|c} n\mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{0} \end{array}\right] =$$

$$\left[\begin{array}{ccccccccccc|ccccccc}
0.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2.96 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 7.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 56.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 153.2 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 153.2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15.0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Solving for $\Delta\mathbf{H}$ and $\Delta\mathbf{Q}$ gives $\Delta\mathbf{H}^{T}$ = [0.38, 1.30, -0.87, -2.28, -2.22, 2.45, 1.07] and $\Delta\mathbf{Q}^{T}$ = [0.94, 0.42, 0.53, -0.13, 0.67, -0.27, -0.14, 0.67, -0.54, -0.40, -0.94].

Step 7: The new nodal heads and pipe flows are then computed by Eqs. 5-45 and 5-46:

$$\mathbf{H}^{(1)} = \mathbf{H}^{(0)} + \Delta\mathbf{H}^{(1)}$$

$$\mathbf{Q}^{(1)} = \mathbf{Q}^{(0)} + \Delta\mathbf{Q}^{(1)}$$

resulting in:

$$\mathbf{H}^{T} = [198.38, 194.30, 194.13, 172.72, 185.78, 192.45, 185.07]$$

and

$$\mathbf{Q}^{T} = [20.94, 9.42, 11.53, 5.87, 6.17, 3.23, 0.36, 1.17, 0.46, 0.60, 7.06]$$

Step 8: The changes during iteration were large so at least one additional iteration is needed. Return to step 2. The results for the remaining three iterations are given in Tables E5-8a and b.

Table E5-8a: Convergence of nodal head with iterations of the pipe method solution.

Iteration	Node 1	2	3	4	5	6	7
1	198	193	195	175	188	190	184
2	198.38	194.30	194.13	172.72	185.78	192.45	185.07
3	198.34	193.95	196.88	170.71	185.96	192.49	184.26
4	198.33	193.89	197.48	170.66	185.96	192.48	184.20
5	198.33	193.89	197.54	170.66	185.96	192.48	184.20

Table E5-8b: Convergence of pipe flow with iterations of the pipe method solution.

Iteration	Pipe 1	2	3	4	5	6	7	8	9	10	Pump
1	20	9	11	6	5.5	3.5	0.5	0.5	1.0	1.0	8.0
2	20.94	9.42	11.53	5.87	6.17	3.23	0.36	1.17	0.46	0.60	7.06
3	21.21	9.79	11.43	6.04	6.07	3.39	0.36	1.07	0.25	0.54	6.88
4	21.27	9.84	11.43	6.03	6.07	3.39	0.36	1.07	0.19	0.54	6.73
5	21.27	9.85	11.43	6.03	6.07	3.39	0.36	1.07	0.19	0.54	6.73

5.4 FIRE FLOW ANALYSIS

A critical factor in water distribution design is the ability to supply adequate pressure during extreme conditions such as fires. This design criterion is based on the fact that fire flow requirements often exceed the normal domestic, industrial, and other demands imposed on the water system. Fire flow is defined as the rate of water flow at a specified residual pressure and for a specified duration that is necessary to control a major fire in a specific structure (AWWA M31). This localized demand can be estimated as discussed in Chapters 4 and 7. Pressure requirements vary between 20 and 40 psi (138 – 275 kPa) during fire and normal demand conditions. Minimum pressures are selected based on local regulations and general guidelines. If the pressures are not acceptable, the design or operations must be modified to meet the pressure requirements.

Four approaches can be used to determine if pressure requirements are satisfied during a fire demand. The first is to define the fire demand at one or more nodes and the demands at other nodes and solve the system of equations for that condition. All hydraulic analysis methods discussed in this chapter assume that the specified nodal demands are satisfied regardless of the pressure that would result in meeting the demand. Although the equations may converge to a valid mathematical solution, the resulting pressures may not be feasible from a design perspective or from a cavitation point of view.

The alternative approaches determine the fire flow that can be provided and insure that the pressure is satisfied at the fire location (Figure 5-5). One technique is to add an emitter with a large discharge coefficient (e.g., 10,000) just downstream of the fire node (Fig. 5-5b). The emitter elevation is set to the desired total head (the actual elevation plus the required pressure head). The numerical solution determines the maximum flow that can be supplied with the nodal head greater than the desired total head and can be compared to the desired fire supply. The emitter discharge coefficient must be sufficiently large so that the computed total head is (nearly) equal to the input junction elevation (elevation plus pressure requirement). A smaller emitter coefficient will not give the maximum achievable fire flow.

The third technique is to add a short pipe to the fire flow node that is connected to a reservoir with total head equal to the elevation plus desired pressure head at the fire flow node (Figure 5-5c). The short pipe should have a negligible head loss (resistance) by using a large diameter and low roughness coefficient. The flow in the pipe to the reservoir equals the demand at the junction node. This configuration results in the same system as the emitter.

In the fourth technique, the mathematical relation between the flow in the connecting pipe and the reservoir head can be expressed in terms of the target pressure and available flow. Under this condition, the fire flow available q_a at a target p_a can be computed from (Boulos et al, 1997):

$$q_a = q_f \left[\frac{p_s - p_a - c(p_f - p_a)}{p_s - p_f} \right]^{\frac{1}{n}} = q_f \left[\frac{H_s - H_a - c(H_f - H_a)}{H_s - H_f} \right]^{\frac{1}{n}} \quad (5\text{-}47)$$

where $c = (q_s / q_f)^n$, q_s designates the static demand at the junction node, p_s is the static pressure, q_f is the normal fire flow demand, p_f is the pressure at the normal fire demand, and n is a flow exponent that is dependent on the head loss expression used. The above expression represents the exact analytical solution of the basic pressure-flow equilibrium relationship and is applicable to any system of consistent units.

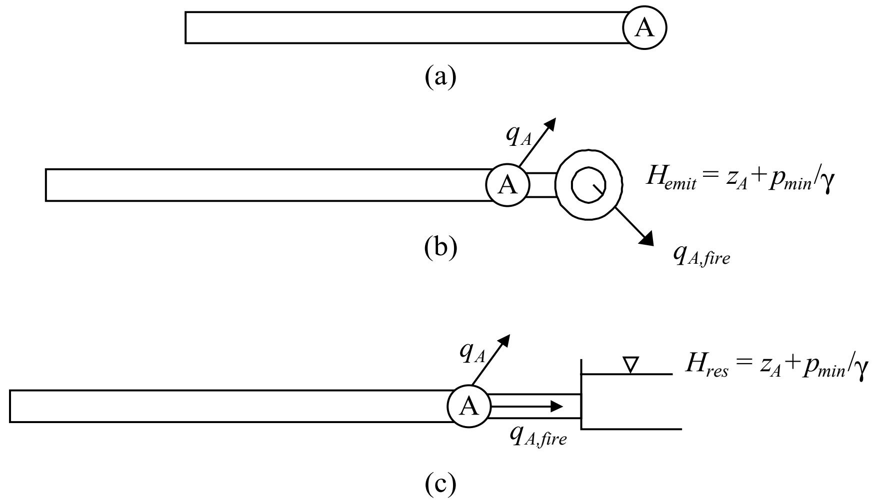


Figure 5-5: Representations to determine fire flow capacity of a system at node A (a). An emitter is added adjacent to the original node (b) and the fire flow is passed through the emitter. Alternatively, a reservoir is attached by a short pipe (c) and flow to the reservoir is the maximum fire flow that can be satisfied while meeting the pressure requirements at node A. In both cases, the total head at the downstream location is set equal to node A's elevation plus the desired pressure head.

Care must be taken when analyzing the results from these configurations. The additional flow coming to the fire flow node causes higher pipe flows (and head losses) throughout the system. Although the fire location's pressure head is maintained by the emitter or reservoir, pressures at other nodes may not be acceptable. For example, if the fire is near the source, downstream nodes at higher elevations may have low (or negative) pressure heads. Thus, the full system results, not only the fire node's pressure head, must be reviewed to determine if the results are satisfactory.

Example 5.9

Problem: Use the emitter approach to determine the maximum flow that can be provided at node 3 under the average demand condition in the network considered in Example 5.4. Assume that the required total head at the node is 290 ft (221 ft node elevation plus 30 psi (69 ft)).

Solution: Table E5-9 lists the emitter flow for a range of emitter coefficients and the computed total heads as computed by a hydraulic analysis model. Note that the 4 cfs nodal demand is supplied in addition to the emitter discharge. In this case, an emitter coefficient over about 50 provides essentially the same result.

Table E5-9: Node data for various emitter coefficients.

Emitter coefficient	Total head (ft)	Emitter discharge (cfs)	Total node withdrawal (cfs)
1	292.01	0.93	4.93
10	290.16	2.60	6.60
20	290.04	2.70	6.70
100	290.00	2.73	6.73

5.5 UNSTEADY FLOW CONDITIONS

Steady flow hydraulic modeling provides a snapshot of the conditions in a distribution system assuming that the hydraulic conditions have reached equilibrium. In general, however, demands vary over time, pump operations are altered or pumps may fail in a sudden event perhaps due to a power failure. These temporal variations cause the pressure and flow distributions to change and can be modeled by three approaches.

The first and most common approach is known as *extended period simulation* (EPS). An EPS is a series of steady state simulations and is an available option in most hydraulic analysis software. An EPS begins with an initial set of tank levels, a given demand distribution and duration and a set of operation decisions. A steady state simulation is completed for the initial set of demands to determine the pressure and flow distribution including flow rates into/out of tanks. Using the tank flow rates and the demand duration, a mass balance calculation is completed to update the tank levels. The new tank levels are then used as the fixed grade node elevations for the next steady state hydraulic analysis and time step. The demands may be changed between time steps. Many hydraulic analysis models allow operation conditions to be altered based upon the hydraulic condition, such as a pump being turned on/off as a function of a tank's water level. The resulting tank flows from a second steady hydraulic analysis are used to update the tank water levels for the third time period. This process is repeated until the entire simulation duration is completed.

The time step between tank level changes in an EPS is typically on the order of hours. This increment is acceptable under most normal operating conditions, particularly for a well-designed system. However, demands may change dramatically at shorter time steps, or operation conditions may be altered rapidly. These changes can cause problems such as low pressure areas that may result in backflows into the system. More detailed modeling may be necessary to detect these problems that would not be apparent when steady state is reached.

Dynamic conditions can be modeled by two approaches. Transient simulation modeling (also known as water hammer analysis or distributed parameter approach) solves the full momentum equation at a time step on the order of seconds or less. Transient analysis is used to examine sudden changes in the system (e.g., pipe or pump failure). The second approach, dynamic simulation (also known as a lumped parameter or rigid column approach), approximates the full momentum equation assuming that the water acts as a rigid column. This simplification permits examination of variations that impact the system on the order of minutes, such as demand changes. A comparison of the various methods can be found in Wood et al (1990). All three simulation approaches are discussed in this section. Transient analysis is introduced in this chapter and presented more fully in Chapter 9.

5.5.1 Extended Period Simulation

In EPS, the only dynamic variables are the tank levels. As noted above, an EPS, also known as a quasi-steady state analysis, consists of a series of steady state simulations with tank levels being updated between steady state analyses. The data requirements for an EPS are discussed in this section with the physical description of the tank, nodal demands as a function of time, and operational controls.

5.5.1.1 Governing Equations

As noted, steady state flow rates are computed in all pipes with the tank levels fixed at their elevations at the beginning of the simulation period. The flow rate into or out of a tank is assumed constant over the duration of the steady state simulation. The new tank levels are computed using the tank mass balance equation (Eq. 2-7):

$$\frac{d}{dt}\int_{CV} d\forall = \frac{d(A_T H_T)}{dt} = A_T\frac{dH_T}{dt} = Q_{T,in} - Q_{T,out} \qquad (5\text{-}48)$$

where $Q_{T,in}$ and $Q_{T,out}$ are the tank inflow and outflow and A_T is the area of the tank and H_T is the tank water level. For a discrete time step of duration Δt and a constant diameter tank, Eq. 5-48 can be written as:

$$A_T\frac{H_{T,t+\Delta t} - H_{T,t}}{\Delta t} = Q_{T,in} - Q_{T,out} \Rightarrow H_{T,t+\Delta t} = H_{T,t} + (Q_{T,in} - Q_{T,out})\frac{\Delta t}{A_T} \qquad (5\text{-}49)$$

where $H_{T,t}$ and $H_{T,t+\Delta t}$ are the tank levels at the beginning and end of the time step Δt, respectively.

The flow rates in Eq. 5-49 are provided by the hydraulic analysis and Δt and $H_{T,t}$ are known. Most tanks are cylindrical and the tank diameter is sufficient to compute the cross sectional area, A_T. In other cases, some models accept a volume versus elevation relationship. The tank storage volume is tracked over time and related to the water level.

In addition, the tank bottom elevation and the range of allowable storages are needed to determine if the tank can accept/supply the inflow/outflow. During a time step a tank may fill or empty to the defined bounds, this time can be computed using Eq. 5-49 with the ending elevation equal to the upper/lower bound. If the tank status does change during a time increment, the period until the change is analyzed with the initial tank levels. The time until the change is computed and a second steady state simulation completed with the tank being isolated by closing the pipe connected to the tank. When a new demand condition is introduced, any isolated tank is checked to see if it should be opened by comparing the tank level and the total head at the node at end of the pipe connecting the tank to the system.

5.5.1.2 Demand Patterns

In addition to physical data about the tanks, the temporal variation of demand must be supplied to the simulation model by a demand pattern. The demand pattern is a set of multipliers that scale the base demand previously established by the user in the junction data. A typical pattern covers a 24-hour cycle to analyze tank level changes during an average day when designing a network or for selecting pump operations on a daily basis. Development of diurnal demand factors is presented in Chapter 7.

As an example, the demand pattern for a node with a base demand of 200 gpm is given in Table 5-1 and plotted in Figure E5-10a. The demand factors or multipliers scale the base demand resulting in the withdrawal pattern shown in the table. The simulation time step is not necessarily equal to the demand pattern time step but the demand pattern step should be an integer multiple of the simulation time step. The average daily demand is often used as the base demand so the average demand multiplier is equal to one.

Table 5-1: Demand pattern multipliers and demands for a node with a 200 gpm base demand.

Time	0-2 hr	2-4	4 -6	6 -8	8-10	10-12	12-14	14-16	16-18	18- 20	20 –22	22 -24
Multiplier	0.75	0.6	0.5	1.1	1.25	1.2	1.1	1.1	1.35	1.2	1.0	0.85
Demand	150	120	100	220	250	240	220	220	270	240	200	170

5.5.1.3 Controls

Utilities often develop operation rules for turning pumps on or off and shutting valves in pipes. Many network models are capable of incorporating this logic as controls within the hydraulic simulation. The model then executes these controls as network conditions dictate providing realistic operations within a single model run.

System controls are often simple (rule-based) logic statements relating pipe or pump operation status (open or closed) to tank levels, nodal pressures or times. For example, turn on pump C (open pump line) if the water level in tank 2 falls below 32 m or shut tank valve (close pipe) between 10 am and 2 pm or if the pressure at node 104 is greater than 40 psi.

More complex if-then statements can activate a status change based on multiple conditions or change operations as a sequence of events occur (Boulos et al 1998). An example is making pipe closure decisions based on time of day and system pressures or tank levels; shut a tank valve (close pipe) if the time between 10 am and 2 pm and the pressure at node 104 is greater than 40 psi else if the clocktime is between 2 and 4 pm shut the pipe if the level in tank 4 is below 140 ft. A second example is to turn on a pump when tank levels are below a given level then off once the tnak reaches an acceptable level; turn pump C on if the water level in tank A is below 82 m and turn pump C off if the water level in tank A is above 88 m.

Example 5.10

Problem: Solve an extended period simulation by modifying the network in Example 5-4 with the data listed in Table E5-10a. Apply the demand pattern in Table 5-1 to all nodes. A cylindrical tank replaces the reservoir 1. The tank diameter and bottom elevation are 75 and 350 ft, respectively, and the minimum and maximum water levels are 10 and 50 ft, respectively. Initially, the tank is empty.

A second pump is added in parallel to pump 1 and connects the network (node 3) to reservoir 2 (Table E5-10a). Only one pump will operate at any time. If the tank is not completely full, pump 1 will be run. When the tank is full, pump 1 will be turned off and the smaller pump 2 will be switched on. This operation will be reversed when the tank is emptying from a full condition.

Given the flows in pipe 1, compute the tank level at 2 and 4 hours. Confirm that the control rules have been properly implemented. System layout is as shown in Figure 5-4a with pumps in parallel connecting reservoir 2 to node 3.

Solution: This network was input to a hydraulic model and the results are summarized in Table E5-10b and Figures E5-10a and b. Table E5-10b lists the tank water surface elevation (WSE) variation over the course of the day. Figure E5-10a is a plot of the tank WSE and demand pattern. At lower demand periods the tank is stable or filling including a rapid rise from midnight to near 4 am while the tank empties during higher demands. Although the demand varies significantly, this pattern maintains a relatively constant pump discharge when pump 1 is operating (Figure E5-10b). A pump should be selected such that it can provide a constant flow near its rated pump capacity to minimize energy and pump maintenance costs.

Table E5-10a: Pipe diameters and pump curves for Example 5.10.

Pipe/pump	Diameter (inches) / pump curve (ft)
1	30
2	24
3	24
4	20
5	20
6	18
7	18
8	18
9	24
10	24
1	$240 - 0.15\,Q^2$
2	$240 - 0.4167\,Q^2$

Table E5-10b: Pipe flow rates during each 2 hour flow period and water surface elevation in tank 1 (at end of period). Note beginning tank water surface elevation was 360 ft.

Time (hours)	Pipe flow (cfs)	Tank WS elevation (ft)
0-2	12.64	380.60
2-4	14.75	400.00
4-6	0	400.00
6-8	-11.86	397.88
8-10	-4.12	391.15
10-12	-2.15	387.66
12-14	0.85	389.05
14-16	0.72	390.23
16-18	-6.07	380.34
18-20	-1.14	378.48
20-22	4.36	385.59
22-24	7.71	398.15

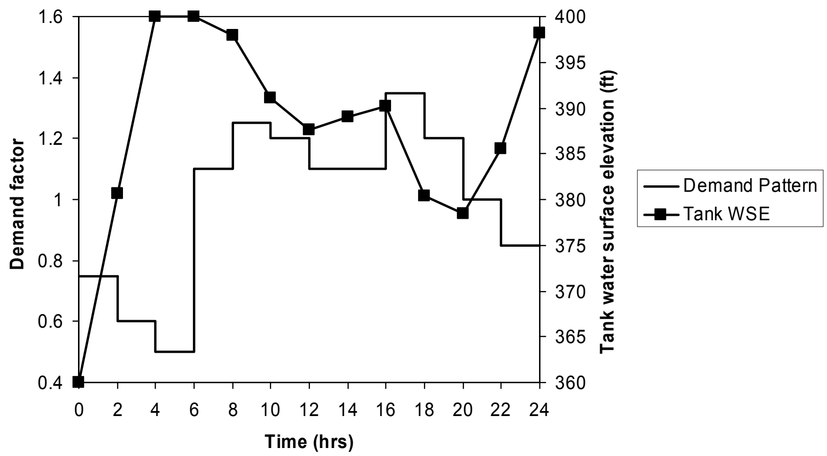


Figure E5-10a: Tank water surface elevation and demand pattern for Example 5.10.

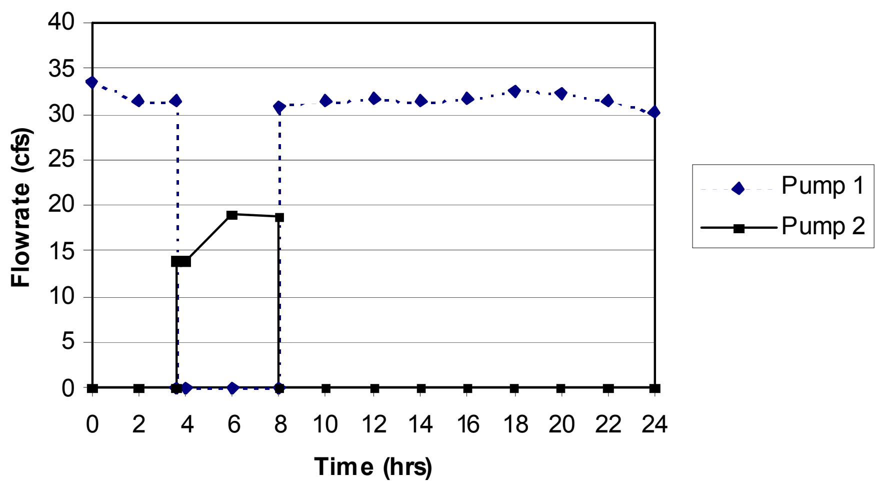


Figure E5-10b: Pump flow rates during EPS simulation in Example 5.10.

To compute the tank level at hour 2 (2 am), we apply Eq. 5-49. Since the demand in the system is low, flow is entering the tank at a rate of 12.64 cfs. The initial tank elevation was 360 ft (the tank level is 10 ft) so the tank elevation at 2 am is computed by:

$$H_{T,t=4am} = H_{T,t=2am} + (Q_{T,in}^{t=12-2} - Q_{T,out}^{12-2})\frac{\Delta t}{A_T}$$

$$= 360\ ft + (12.64\ ft^3/s)\frac{7200\,s}{4418\ ft^2} = 360 + 20.60 = 380.60\,ft$$

The tank level is still with the acceptable range.

A similar calculation is completed for time 4 hours (4 am). Again the network demand is low so the tank level is rising. Substituting the inflow and starting elevation in Eq. 5-49 gives:

$$H_{T,t=4\,am} = H_{T,t=2\,am} + (Q_{T,in}^{2-4am} - Q_{T,out}^{2-4\,am})\frac{\Delta t}{A_T}$$
$$= 380.60 + (14.75)\frac{7200}{4418} = 380 + 24.04 = 404.04\,ft$$

This water level exceeds the maximum water level before 4 am. Therefore, pipe 1 must be closed to stop water from entering the tank before it overtops. The closure time is computed using the tank mass balance equation with the unknown as the closure time or:

$$H_{T,2\,am+\Delta t} = 400 = H_{T,t=2\,am} + (Q_{T,in}^{2\,am} - Q_{T,out}^{2\,am})\frac{\Delta t}{A_T}$$
$$= 380.60 + (14.75)\frac{\Delta t}{4418} = 380.6 + 0.003339\,\Delta t$$
$$\Rightarrow \Delta t = 5811\,s \Rightarrow t = 2 + \frac{5811}{3600} = 3.61\,hr = 3\,hr\;\;37\,min$$

At about one and a half hours after the demand change, the tank line is closed. Since the tank is full, the control rules are activated. By the defined rule, when the tank was filled the larger pump (i.e., pump 1) is shut down and the smaller pump (i.e., pump 2) is switched on (Figure E5-10b). This operation is maintained until 6 am (Figure E5-10b) when the demand begins to increase and the water is withdrawn from the tank (Figures E5-10a).

5.5.2 Dynamic Simulation

During a steady state or EPS simulation, demand loading and operating conditions are assumed to change instantaneously and steady state is reached immediately after a change occurs. In other cases, the transition between the various hydraulic conditions may be important. Dynamic simulation (lumped parameter approach), considers gradually varied flow and slow moving transients under the assumption that water acts as a rigid-column. A rigid-water-column is assumed to have constant density that is not affected by the changes in water pressure. Full transient analysis (distributed parameter approach) considers an elastic-water-column in which water density may be variable (Chapter 9).

5.5.2.1 Governing Equations

In a dynamic state, pressure waves and forces cause flow variations. As such, the governing equations are conservation of mass and momentum, rather than conservation of mass and energy. As under steady state conditions, conservation of mass is flow balance at a node. Conservation of momentum is applied to a pipe element (Figure 5-6) in which the net force on the fluid equals the time rate of change of momentum in the element or:

$$\sum F = F_1 - F_2 - F_f = \frac{d(mV)}{dt} = m\,a \qquad (5\text{-}50)$$

where F_1 and F_2 are the forces at the end of the element due to the total head of the fluid and F_f is the force due to friction. This balances with the rate of change of momentum of the fluid in the element that has mass, m, and velocity, V. This rate of change is equal to the mass times the acceleration, a, of the fluid mass. Thus, Eq. 5-50 is Newton's Second Law ($F = ma$).

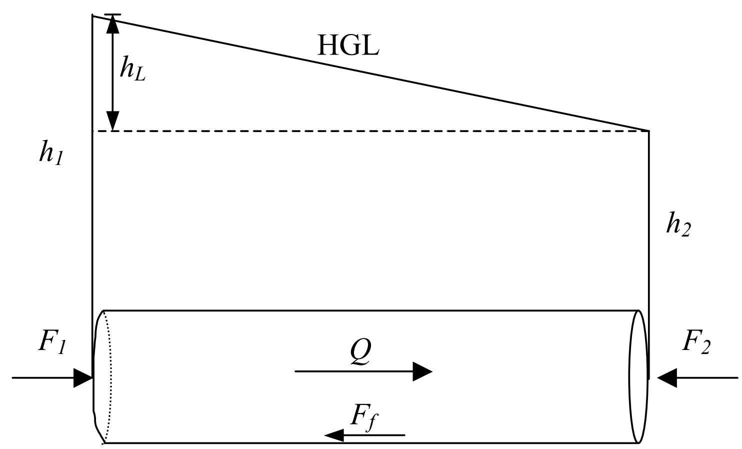


Figure 5-6: Force balance on a pipe segment with length, *L*, and cross section area, *A*.

The end forces are equal to the force due to the pressure and gravity or:

$$F_1 = \gamma A\left[\frac{p_1}{\gamma} + z_1\right] = \gamma A H_1 \qquad (5\text{-}51)$$

where A is the cross sectional pipe area and γ is the specific weight of water. A similar term is written for the left side. The friction force is the force caused by the energy loss in the pipe element, h_L, or:

$$F_f = \gamma A h_L \tag{5-52}$$

The rate of change of momentum can be expanded to consist of similar terms as:

$$\frac{d(mV)}{dt} = \frac{d(\rho \forall V)}{dt} = \frac{d\left(\frac{\gamma}{g} ALV\right)}{dt} = \frac{\gamma L}{g}\frac{d(AV)}{dt} = \frac{\gamma L}{g}\frac{dQ}{dt} \tag{5-53}$$

Assuming that the only head loss is due to pipe friction and substituting terms in Eq. 5-50 results in a relationship similar to conservation of energy except that the right hand side is not equal to zero during unsteady conditions (Wood et al, 1990):

$$H_1 - H_2 - K\lfloor Q \rfloor^n = \frac{L}{gA}\frac{dQ}{dt} \tag{5-54}$$

A generalized form presented by Sevuk (1979) is:

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + K_L g A\lfloor Q \rfloor^n = 0 \tag{5-55}$$

where the head difference in Eq. 5-54 becomes the partial of H with respect to x and the coefficient, K_L, equals K/L or a unit length coefficient.

5.5.2.2 Solution Methods for Gradually Varied Flow

Eqs. 5-54 and 5-55 can be written for a single pipe or a set of pipes for a closed or pseudo-loop. These equations can be combined with conservation of mass to form several different sets of equations that can be solved for the total heads and the pipe flows. Formulations parallel those for steady state and vary with the form for conservation of energy.

Two general solution approaches are taken. The first integrates the flow equations and approximates the friction loss term by the ending time conditions. The result is a mathematical structure that is similar to steady state modeling but with an additional term in the energy balance from the momentum change (Eq. 5-53). Models following this approach are presented with some detail below. The second approach is to numerically solve the generalized form of conservation of energy (Eq. 5-55) as a partial differential equation (Sevuk, 1979) or as an ordinary differential equation (Onizuka, 1986; Shimada, 1989; and Dunlop, 1999). These approaches are not presented here in detail.

5.5.2.21 Simultaneous Loop Corrections. Wood et al (1990) formulated the loop equations (Simultaneous Loop Adjustment Method) in terms of gradually-varied flow. They compared their results with water hammer transient models and showed that the lumped model was acceptable in some transient cases. The algorithm as presented requires fixed nodal demands so only physical system changes such as valve closure and source head variations can be examined.

In this formulation, a loop equation is written for each closed and pseudo-loop of the form of Eq. 5-54:

$$\overline{H}_A - \overline{H}_B - \sum_{i \in lpath} \overline{h}_{L,i} = \sum_{i \in lpath} \frac{L_i}{g\,A_i} \frac{dQ_i}{dt} \tag{5-56}$$

where the $\overline{H}_A$ and $\overline{H}_B$ are the average total nodal heads at the ends of the loop and Q_i and $\overline{h}_{L,i}$ are the flow rate and average head loss in component *i*.

These ordinary differential equations are written in a discrete form in terms of the change in flow rate over a short time increment, Δt, or:

$$\overline{H}_A - \overline{H}_B - \sum_{i \in lpath} \overline{h}_{L,i} = \sum_{i \in lpath} \frac{L_i}{g\,A_i} \frac{\Delta Q_i}{\Delta t} \tag{5-57}$$

The average boundary head values are assumed to be known or, for a closed loop, to be equal (i.e., $\overline{H}_A = \overline{H}_B$). The head loss is taken as the initial head loss plus the one-half of the change in head loss over the incremental time step or:

$$\overline{h}_{L,i} = h_{L,i}^t + \sum_{lp \in ncp(i)} \frac{\partial h_{L,i}}{\partial Q_{lp}} \frac{\Delta Q_{lp}}{2} \tag{5-58}$$

where the gradients are the changes in head loss due to a change in flow and are evaluated at time *t*. ΔQ_{lp} is the change of flow during the time step in loop *lp* that contains pipe *i*. It was assumed that the half of the flow change will provide a good approximation of the average head loss for the time period. For a closed loop, the equation becomes:

$$-\sum_{i \in lpath} \left(h_{L,i}^t + \sum_{lp \in ncp(i)} \frac{\partial h_{L,i}}{\partial Q_{lp}} \frac{\Delta Q_{lp}}{2} \right) = \sum_{i \in lpath} \left(\sum_{lp \in ncp(i)} \frac{L_i}{g\,A_i} \frac{\Delta Q_{lp}}{\Delta t} \right) \tag{5-59}$$

All values are known from physical data or from the flow distribution at time *t*. One equation is written for each of *lloop* loops in terms of the *lloop* flow changes. The system of *lloop* equations is linear in terms of the *lloop* unknown ΔQ_l's due to the average head loss change assumption. Once the loop flow changes are computed, the flows are updated for the next time step by:

$$Q_i^{t+\Delta t} = Q_i^t \pm \sum_{lp \in ncp(i)} \Delta Q_{lp} \qquad (5\text{-}60)$$

The algorithm then proceeds to the next time step and computes the flow change for that time step.

If variable nodal demands were introduced, mass balance at each node must be preserved. An iterative approach would likely be necessary that begins with a solution that satisfies conservation of mass at the next time step or conservation of mass could be explicitly included in a formation like that in the previous section.

Onizuka (1986) solved the loop equations with time varying nodal demands by formulating and solving the set of ordinary differential equations (Eq. 5-54) with conservation of mass using the Runge-Kutta fourth order scheme.

5.5.2.2.2 Node-Loop Equations. Holloway (1985) and Islam and Chaudhry (1998) structured the equations in a node-loop formulation resulting in *npipe* equations for the *npipe* flow rates. For a closed loop, the nodal heads in Eq. 5-54 are the total pressure at the same point (beginning and end of the loop). Thus, it becomes:

$$-\sum_{l \in lloop} K_l\, Q_l^n = \sum_{l \in lloop} \frac{\gamma L}{g} \frac{(Q_l^{t+\Delta t} - Q_l^t)}{\Delta t} \qquad (5\text{-}61)$$

This equation is solved forward in time for the values of $Q^{t+\Delta t}$ by separating variables and integrating over time:

$$\int_t^{t+\Delta t} -\left[\sum_{l \in lloop} K_l\, Q_l^n\right] dt = \int_{Q^t}^{Q^{t+\Delta t}} \sum_{l \in lloop} \frac{L}{g\,A}\, dQ = \sum_{l \in lloop} \frac{L}{g\,A} (Q_l^{t+\Delta t} - Q_l^t) \qquad (5\text{-}62)$$

Holloway (1985) suggested that the friction loss term on the left-hand side may be approximated in several ways:

$$K\, Q^{t+\Delta t} \left|Q^t\right|^{n-1} \Delta t \qquad (5\text{-}63)$$

$$K\left[(Q^{t+\Delta t} + Q^t)\left|Q^{t+\Delta t} + Q^t\right|^{n-1} / 2^n\right]\Delta t \qquad (5\text{-}64)$$

$$K\left[\left(Q^{t+\Delta t}\left|Q^{t+\Delta t}\right|^{n-1}+Q^{t}\left|Q^{t}\right|^{n-1}\right)/2^{n}\right]\Delta t \tag{5-65}$$

The first approximation (Eq. 5-63) maintains linearity with respect to the unknowns $Q^{t+\Delta t}$. Holloway compared this integration approximation with the two nonlinear forms (Eq. 5-64 and 5-65) and demonstrated equivalent results. Substituting Eq. 5-63 in Eq. 5-62 and manipulating gives:

$$\sum_{l\in lloop}\frac{L}{g\,A}Q_l^{t+\Delta t}-K\,Q_l^{t+\Delta t}\left|Q_l^{t}\right|^{n-1}\Delta t=\sum_{l\in lloop}\frac{L}{g\,A}Q_l^{t} \tag{5-66}$$

Eq. 5-66 is written for each closed and pseudo-loop as in modified linear theory for steady flow. The node equations written in terms of the pipe flows at time $t + \Delta t$ provide *nnode* additional equations. The set of linear equations are solved for the flow rates at time $t + \Delta t$. Once those flows have been calculated, a new set of linear equations for the next step can be formulated. Time varying demands are introduced in the node equations. The time step can affect the convergence of the results but no general rule was provided.

Shimada (1989, 1992) formulated the node-loop equations as ordinary differential equations (ODE) and updated flow directly using more stable ODE methods (Runge-Kutta fifth order with a variable time step and the Kaps-Rentrop semi-implicit scheme)). These solution algorithms are better equipped to solve so-called stiff ODEs such as Eq. 5-54, particularly as the system changes (e.g., valve closures) are rapid. Shimada (1989) validated the model with comparisons with full transient (water hammer) solutions.

5.5.2.2.3 Pipe-Node Equations. Ahmed and Lansey (1999) formulated the pipe-node equations (hybrid or gradient method) in an integral form similar to Holloway (1985). The advantage of this method is that same advantage that the pipe-node formulation has in steady state analysis; loops do not need to be identified that allows larger systems to be more easily analyzed.

Like the steady state formulation, conservation of energy is written for a single pipe (or general component).

$$H_{il}-H_{jl}-K_l\left\lfloor Q_l\right\rfloor^{n}=\frac{L_l}{g\,A_l}\frac{dQ_l}{dt} \tag{5-67}$$

where H_{il} and H_{jl} are the piezometric heads at the upstream node (i) and downstream node (j) of pipe l.

The time derivative was approximated by an explicit backward difference that results in:

$$-K_l\lfloor Q_l^{t+\Delta t}\rfloor^n \Delta t-(L_l/gA_l)Q_l^{t+\Delta t}=-(L_l/gA_l)Q_l^t-(H_i^{t+\Delta t}-H_j^{t+\Delta t})\Delta t \quad (5\text{-}68)$$

Combined with the nodal balance equations, a nonlinear system of pipe equations is formed that can be solved for the pipe flows and nodal heads using a gradient algorithm based method. Several alternative schemes to approximate the head loss can be applied including a linear formulation. This method and Holloway's have only been documented on slowly varying conditions. Additional testing of these approaches is necessary.

Dunlop (1999) solved the ODE formulation of the pipe-node equations using Gear's method. In addition, he demonstrated that pressure dependent demands could be determined simultaneously within the method and that a range of components could be analyzed. Sevuk (1979) posed the pipe-node formulation with partial differential equations and used an implicit scheme for solving the resulting quasi-linear hyperbolic PDEs.

5.5.3 Water Hammer Simulation

The dynamic simulation that we have just considered involves the acceleration and deceleration of a fluid mass. It is powerful solution technique in that, unlike the EPS approach, it explicitly accounts for the inertia in the water column. However, the limitation with the rigid column approach also lies in its basic assumption that the length of water in each pipe acts as single mass with a uniform velocity in each conduit. The question is how such a coordination of a whole mass of fluid is achieved: what law actually specifies that what happens at one end of a pipe at one instant must also happen at the same instant at the other?

In fact, when a column of fluid is rapidly accelerated, a small amount of mass does accumulate within the pipe, thus mobilizing not only inertia terms, but also fluid compressibility. It is this compressibility that sends a pressure signal through the pipe, and informs the whole column of the changes taking place within it. The price of this approach is that we must now discretize the pipe, breaking it up into pieces that explicitly permit the propagation of a pressure signal. This approach has greater computation demands, but also permits a greater degree of pressure and velocity variation within the pipe. These compressible or water hammer models often result in more realistic unsteady flow calculations than those associated with the rigid model. This interesting and important topic is developed in more detail in Chapter 9.

5.6 PRESSURE DRIVEN ANALYSIS

The solution methods described to this point can be described as demand driven analyses (DDA). In these formulations, the demand at nodes is predefined and is satisfied regardless of the nodal pressure; even when the pressure is negative. In most cases, this result is acceptable under normal operating conditions as pressure defines an acceptable state and if the pressure is less than an allowable value, the system must be changed.

To understand how the system will operate under low pressure conditions, a pressure driven analysis (PDA) can be posed. In a PDA, the nodal withdrawal is dependent upon the pressure at the node. Demand locations were represented by emitters (Reddy and Elango, 1989). As seen in Section 5.4, the withdrawal is related to the nodal pressure by the orifice equation:

$$q = C_{emit}\ p^{\varphi} \tag{5-69}$$

The problem with direct application of this relationship is that the discharge is not bounded above. If the pressure is high, the discharge may be higher than the consumer demand. Therefore, the discharge should be limited to the flow provided when the demand is reached. At the lower bound, the withdrawal equals zero if the pressure is equal to less than 0. Some researchers suggest that demand will be modified with the pressure beyond what is expected from Eq. 5-69. For example, in a reliability analysis Wagner et al (1988) proposed:

$$q = q_{dem}\left(\frac{p_{avl} - p_{\min}}{p_{des} - p_{\min}}\right)^{\alpha} \tag{5-70}$$

where q_{dem} is the consumer demand, p_{avl}, p_{des}, and p_{min} are the available, design and minimum allowable pressures at the node, and α is a coefficient typically in the range of 0.5 to 0.7. These bound relationships and new function forms can be introduced in the steady state formulations presented in Section 5.3.

Problems:

Problem 1. A pipeline consists of four pipes in series with physical data listed below. Flow only exits through the most downstream pipe (pipe D). For a flow rate of 0.5 m^3/s,

a) Compute the equivalent head loss coefficient
b) Compute the total head loss occurring through four pipes

Table P5-1: Data for series pipes (including *K* value).

Pipe	Diameter (cm)	Length (m)	Friction factor	$K = f\,L/(D\,2g)$	h_L (m)
A	100	150	0.021	0.1606	0.040
B	90	200	0.024	0.2718	0.068
C	80	100	0.027	0.1720	0.043
D	70	125	0.031	0.2821	0.071

Problem 2. The same four pipes in Prob. 5-1 are now placed in parallel. For a head loss from the upstream to downstream nodes of 0.6 m,

a) Compute the equivalent head loss coefficient
b) Compute the total flow occurring through the four pipes

Table P5-2: Data for series pipes (including *K* value).

Pipe	Diameter (cm)	Length (m)	Friction factor	$K = f\,L/(D\,2g)$	Q (m^3/s)
A	100	150	0.021	0.1606	1.93
B	90	200	0.024	0.2718	1.49
C	80	100	0.027	0.1720	1.87
D	70	125	0.031	0.2821	1.46

Problem 3. Using the data below, fill in the following blanks in the output tables (Tables E5-3a and b). The network layout is shown in Figure P5-3.

a) Pipe flow in pipe 10 (flow is from the reservoir (node 8) to node 7)
b) Pipe flow in pipe 4
c) Friction factor in pipe 1
d) Velocity in pipe 7
e) Pump flow rate
f) Confirm the energy balance in the loop containing node 1
g) Confirm the energy balance over the pseudo-loop
h) Demand at node 3
i) Total and pressure heads for node 6

Table P5-3a: Junction data for Problem 5-3.

Node ID	Elevation (m)	Demand (m^3/hr)	Head (m)	Pressure (m)
Junc 1	80	450	103.66	23.66
Junc 2	80	150	105.85	25.85
Junc 3	75		102.32	27.32
Junc 4	100	187.5	120.98	20.98
Junc 5	80	750	100.7	20.7
Junc 6	80	187.5		
Junc 7	150	187.5	170.1	20.1
Resvr 8	210	-1512.5	210.0	0.
Resvr 9	10	-775	10.0	0.

Problem 4.

a) Set up the network in Problem 5-3 in a network solver. Use the Darcy-Weisbach equation for concrete pipes; e = 1.5 mm.
b) Simulate the system 1.5 times the original nodal demands. Examine the friction factor and the nodes with minimum total head.

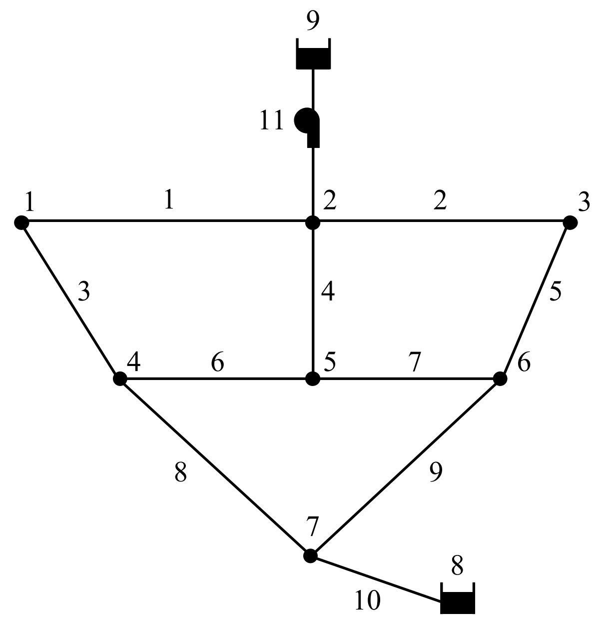


Figure P5-3: Problem 5-3 network.

Table P5-3b: Link data for Problem 5-3. (Note negative flows denote flow direction is opposite of assumed direction as defined by nodes.)

Link ID	Upst. Node	Downst. Node	*L* (m)	*D* (mm)	*e* (mm)	Flow (cmh)	Velocity (m/s)	Unit Headloss (m/km)	Friction Factor
Pipe 1	1	2	1000	300	1.5	-163.97	0.64	2.19	
Pipe 2	2	3	1000	300	1.5	208.64	0.82	3.53	0.031
Pipe 3	1	4	1000	250	1.5	-286.03	1.62	17.32	0.032
Pipe 4	2	5	1000	300	1.5		0.99	5.15	0.031
Pipe 5	3	6	1000	250	1.5	-166.36	0.94	5.89	0.033
Pipe 6	4	5	1000	250	1.5	309.62	1.75	20.28	0.032
Pipe 7	5	6	1000	250	1.5	-188		7.51	0.033
Pipe 8	4	7	1000	300	1.5	-783.15	3.08	49.12	0.031
Pipe 9	7	6	1000	250	1.5	541.86	3.07	61.88	0.032
Pipe 10	7	8	1000	400	1.5		3.34	39.9	0.028
Pump 11	9	2						-95.85	

Table P5-3c: Pump curve.

Pump Head (m)	Pump Flow (m^3/h)
240	0
180	500
0	1000
$h_p = 240 - 0.00024\,Q^2$	

Problem 5.

a) Set up the network in Problem 5-3 in a network solver. Use the Hazen-Williams equation with a C_{HW} value of 120.
b) Simulate the system with 1.5 times the original nodal demands. Examine the friction factor and the nodes with minimum total head.

Problem 6. Input the system with the layout and assumed flow directions given in Figure P5-6. The results shown in the following problems may not exactly match the results from other computer models since slight differences in Hazen-Williams exponent and conversions from input units to the base unit of calculation may differ. The deviations, however, should be minor and care should be taken to confirm that the input data is correct (in particular minor loss coefficients).

Table P5-6a: Pipe data for Problem 5-6 (C_{HW} value for all pipes is 130).

Pipe	Length (ft)	Diameter (in)	Minor loss coeff.
1	1000	8	0
2	1000	8	0
3	1300	8	0
4	1000	10	0
5	1200	10	0
6	1400	10	0
7	1600	10	0
8	2500	10	0
9	2000	12	0
10	900	10	0
11	900	12	1
12	1200	8	0
13	1300	10	0
14	1000	10	2
15	500	12	5

Table P5-6b: Pump data for Problem 5-6.

Pump head (ft)	Pump discharge (cfs)
178	0
150	5
50	10

* $h_p = 178 - 0.8214\,Q^{2.19}$

Problem 7. In the network for Problem 5-6, add a parallel pipe to pipe 8 with the length, diameter and roughness of 2500 ft, 12 in, and 130. Also a second identical pump in parallel to pump 16 and simulate the system with the same demands as Problem 5-6.

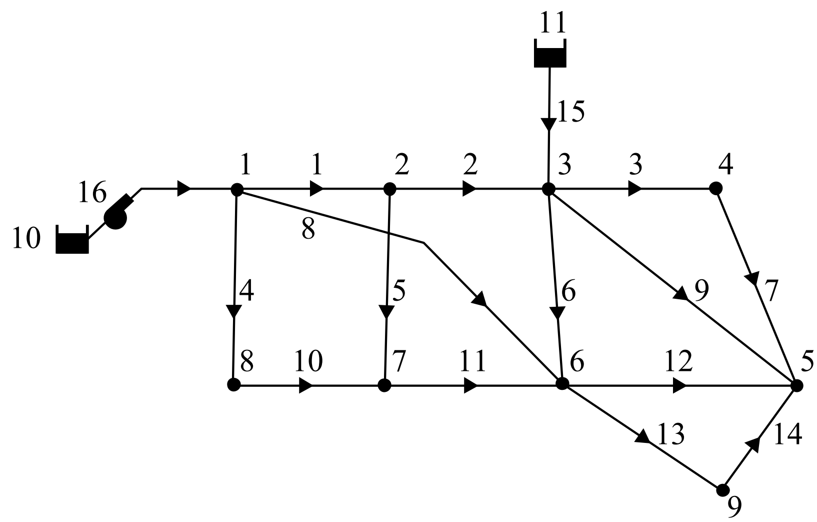


Figure P5-6: Problem 5-6 network layout.

Table P5-6c: Node data for Problem 5-6.

Node	Demand (cfs)
1	0
2	0.5
3	0
4	0.5
5	2
6	1
7	0.5
8	1.5
9	2
10	-4.82
11	-3.18

Problem 8. In the Problem 5-6 network, install pressure reducing valves in pipes 13 and 14 just downstream of nodes 6 and 5, respectively. Model the system with the PRV settings at 50 psi and at 65 psi.

Problem 9. For the Problem 5-6 network, determine maximum fire flow that can be supplied with a pressure of 30 psi at node 5 for the single pump system.

Problem 10. Using the network from Problem 5-7, change reservoir 11 to a tank with a diameter of 50 ft and minimum and maximum levels of 135 and

175 ft, respectively. The elevation at this location is 0 ft and the initial water level is 150 ft. Simulate a 24 hour EPS in four hour increments using the demand pattern given in Table P5-10a. Verify the changes in water level over time.

Table P5-10a: Demand pattern for Problem 5-10.

Time (hrs)	Demand Multiplier
0-4	0.75
4-8	0.9
8-12	1.1
12-16	1.25
16-20	1.0
20-24	1.0

Problem 11. Using the network developed for Problem 5-10, add controls to operate the pumps based on the tank levels as follows. If the tank level is above 155 ft, no pumps are on. If the tank level is below 145 ft, both pumps are operating and if the level is between 145 and 155 ft, one pump is on and the other is off.

Problem 12. Determine the flow distribution for the network shown in Figure P5-3 using the Hardy Cross method.

Apply the Darcy-Weisbach equation with the friction factor in all pipes equal to 0.032 which assumes fully turbulent conditions in all pipes. A constant f simplifies the analysis and avoids computing the gradients of the friction factor with respect to the flow rate (e.g., with the Swamee-Jain equation). The resulting K values are given in Table P5-12a.

The nodal demands and pipe and pump characteristics are listed in Tables P5-3a-c. Flow directions are assumed on the Figure P5-12 and values are given in Table P5-12a. Note that pipes 3, 5, and 10 flow in the opposite direction of that shown in Figure P5-12 so the negative assumptions are correct to achieve mass balance. Although the flow directions for those pipes are reversed, the signs on the flow rates will be in the proper direction (see below). The purpose in this problem is to demonstrate how to handle an incorrect flow direction assumption.

Problem 13. Solve Problem 5-12 using the simultaneous loop equation method using the same initial solution.

Problem 14. Solve Problem 5-12 using the loop-node equation method using the same initial pipe flow rates.

Table P5-12a: Initial flow rates, head loss coefficients, head losses and derivative terms for Problem 5-3 network.

Pipe	1	2	3	4	5	6	7	8	9	10	P
Flow (cmh)	200	287.5	-250	250	-87.5	287.5	212.5	725	487.5	-1400	887.5
Flow (cms)	0.0556	0.0799	-0.0694	0.0694	-0.0243	0.0799	0.059	0.2014	0.1354	-0.389	0.2465
K (for cms)	1088	1088	2708	1088	2708	2708	2708	1088	2708	258	**
h_L	3.36	6.94	13.06	5.25	1.60	17.27	9.43	44.13	49.65	39.05	50.96
nh_L/Q	120.9	173.8	376.0	151.1	131.6	432.5	319.6	438.3	733.3	200.8	1533.6*

* Absolute value of the derivative of the pump equation with respect to pump discharge
** The pump equation for Q in cms is $h_p = 240 - 3110\ Q_p^2$

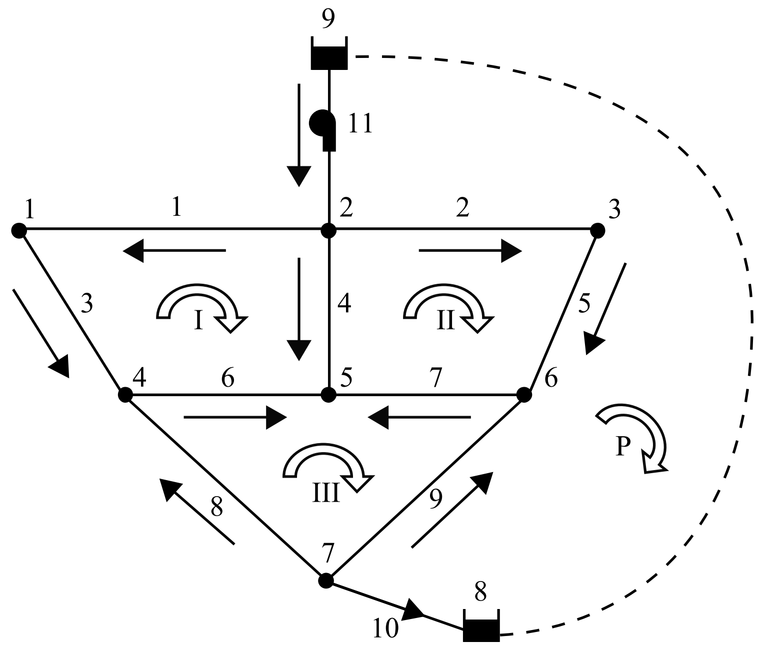


Figure P5-12: Problem 5-12 network with assumed flow directions. Note that the flow directions for pipes 3, 5, and 10 are assumed incorrectly.

Problem 15. Solve Problem 5-12 using the node equation method. Use the following initial solution. The pipe flows corresponding to the assumed nodal heads are found by the D-W equation (Eq. 5-26) and are listed in Table P5-15b. With the negative signs these flows follow the correct flow distribution.

Problem 16. Solve Problem 5-12 using the pipe equation method. Use the initial solutions for flow rates from Problem 5-12 and the nodal heads from Problem 5-15.

Table P5-15a: Initial nodal heads for node equation solution.

Node	1	2	3	4	5	6	7
Total head (m)	103.66	105.85	102.32	120.98	100.7	108.22	170.1
Nodal demands (m^3/hr)	450	150	375	187.5	750	187.5	1875
Nodal demand (m^3/s)	0.125	0.042	0.104	0.052	0.208	0.052	0.052

Table P5-15b: Computed flow rates for the given initial set of nodal heads.

Node	1	2	3	4	5	6	7	8	9	10	Pump
Flow (m^3/s)	0.0449	0.057	-0.08	0.0688	-0.0467	0.0865	0.0527	0.2125	0.1512	-0.393	0.2153
Flow (m^3/h)	161.5	205.0	-287.9	247.7	-168.1	311.6	189.7	764.9	544.2	-1415.2	775.0

<u>**Solutions:**</u>

1. a) To compute the equivalent head loss coefficient for pipes in series the head loss coefficients, K_l, are summed or by Eq. 5-2:

$$K_{eq}^{s} = \sum_{l=1}^{lpath} K_l$$

The K for each pipe is computed in the last column of Table P5-1 with n = 2. The equivalent K is then:

$$K_{eq}^{s} = \sum_{l=1}^{lpath} K_l = K_1 + K_2 + K_3 + K_4$$
$$= 0.1606 + 0.2718 + 0.1720 + 0.2821 = 0.8865$$

b) The total head loss is:

$$h_L = K_{eq}^{s}\ Q = 0.8865\ (0.5)^2 = 0.22\ m$$

This value is equal to the sum of the head losses for the individual segments (Table P5-1).

2. a) To compute the equivalent head loss coefficient for pipes in parallel the inverse head loss coefficients, K_l, are summed or by Eq. 5-2:

$$\left(\frac{1}{K_{eq}^{s}}\right)^{1/n} = \sum_{l=1}^{lpath}\left(\frac{1}{K_l}\right)^{1/n}$$

The K for each pipe is computed in the last column of Table P5-1. The equivalent K is then:

$$\left(\frac{1}{K_{eq}^{s}}\right)^{1/n} = \sum_{l=1}^{lpath}\left(\frac{1}{K_l}\right)^{1/n} = \left(\frac{1}{K_1}\right)^{1/n} + \left(\frac{1}{K_2}\right)^{1/n} + \left(\frac{1}{K_3}\right)^{1/n} + \left(\frac{1}{K_4}\right)^{1/n}$$

$$= \left(\frac{1}{0.1606}\right)^{1/n} + \left(\frac{1}{0.2718}\right)^{1/n} + \left(\frac{1}{0.1720}\right)^{1/n} + \left(\frac{1}{0.2821}\right)^{1/n} = 8.707$$

$$\Rightarrow \left(\frac{1}{K_{eq}^{s}}\right)^{1/n} = 8.707 \Rightarrow K_{eq}^{s} = 0.0132$$

b) The total flow rate is then:

$$h_L = K_{eq}^{s}\, Q = 0.0132\, Q^2 = 0.6\, m \Rightarrow Q = 6.74\, m^3 / s$$

This value is equal to the sum of the flow in the individual pipes (Table P5-2).

3. a) Pipe flow in pipe 10 is found by conservation of mass:

$$Q = A\,V = \frac{\pi\, D^2}{4} V = \frac{\pi\,(0.400)^2}{4}(3.34\, m/s)\left(3600\, s/hr\right) = 1512.5\, m^3 / hr$$

In the output file, $Q_{10} = -1512.5\ m^3/h$.

b) Pipe flow in pipe 4

Use the nodal mass balance for node 5. In this equation, Q_7 has a negative sign since that flow is assumed to be from node 5 to node 6 or:

$$Q_4 + Q_6 - Q_7 = q_5 = Q_4 + 309.62 - (-188) = 750 \Rightarrow Q_4 = 252.38\ m^3/h$$

The negative sign in the model output for node 7 implies that the flow is actually from node 6 to node 5.

c) Friction factor in pipe 1

The friction factor can be determined from the Darcy-Weisbach equation or:

$$h_f = f\,\frac{L\,V^2}{D\,2g} = 2.19(1000/1000) = f\,\frac{1000\,(0.64)^2}{(0.3)\,2\,(9.81)} \Rightarrow f = 0.031$$

d) Velocity in pipe 7

By continuity:

$$Q_7 = A_7 V_7 = 188\,(1/3600) = \frac{\pi D_7{}^2}{4} V_7 = \frac{\pi\,(0.25)^2}{4} V_7 \Rightarrow V_7 = 1.06\ m/s$$

e) Pump flow rate

Since the pump head is given as (with a positive sign since the negative head loss is a gain) we can substitute in the pump equation for Q:

$$H_p = 240 - 0.00024\,Q^2 = 95.85 \Rightarrow Q = 775\ m^3/h$$

f) Confirm the energy balance in the loop containing node 1

This loop consists of four pipes (1, 4, 6, and 3). The sum of the head losses in this loop should equal zero. Negative signs are given to flows in the counter-clockwise direction:

$$h_{L,1} + h_{L,4} + h_{L,6} + h_{L,3} = -2.19\,(1000/1000) + 5.15\,(1000/1000) - 20.28\,(1000/1000) + 17.32\,(1000/1000) = 0$$

The sum equals zero confirming conservation of energy.

g) Confirm the energy balance across the pseudo-loop.

The pseudo-loop begins at node 9 through the pump and pipes 4, 7, 9 and 10 to node 8. Energy losses are given negative signs if the flow is toward node 8 and positive (a gain if away from node 8) or:

$$H_9 - h_p + h_{L,4} + h_{L,7} + h_{L,9} + h_{L,10} = H_8 = 10 - (-95.85) - 5.15\,(1000/1000) + 7.51\,(1000/1000) + 61.88\,(1000/1000) + 39/9\,(1000/1000) = 210\ m$$

The head gains/losses balance to the difference in head between the two reservoirs.

h) Compute the nodal demand for node 3.

By conservation of mass at the node:

$$Q_2 + Q_5 = q_3 = 208.64 - (-166.36) = 375\ m^3/h$$

i)Total and pressure heads for node 6.

Starting at the known total head at node 7, we can substrate the head losses in the connecting pipe 9 to find the total head at node 6 or:

$$H_7 - h_{L,9} = 170.1 - 61.88 = 108.22\ m = H_6$$

The pressure head for node 6 from definition of the total head and the given elevation:

$$H_6 = z_6 + p_6/\gamma = 108.22 = 80 + p_6/\gamma \Rightarrow p_6/\gamma = 28.22\ m$$
$$\Rightarrow p_6 = 28.22\ (9810) = 276.8\text{ kPa}$$

4. a) The minimum total head was 100.7 m at node 5. The friction factors were near 0.031 except for pipe 10 ($f = 0.028$).

b) The minimum total head was located at node 4 and many nodes had negative total heads. The friction factors were in the range of 0.031. Negative heads are not realistic but demonstrate that the system of equations can be solved mathematically, even for this condition.

5. a) The minimum total head was at node 5 but the head was 135.72 m. Although both the friction factor and C_{HW} were for the same pipe type, the equivalent friction factors were all less than 0.023 with some as low as 0.018 for this case compared to 0.03 for the D-W equation.

Note the differences in heads were affected by a different flow distribution and more flow from the pump (with less head).

b) The minimum total head also occurred at junction 5 and was less than 40 m. The velocities in this system were excessive and resulted in negative pressures. The equivalent friction factors, f, were still less than 0.023 and most were less than 0.018. The impact was significantly less than when using the D-W equation.

6. Tables P5-6d and e present results from the numerical simulation. The pump is operating near the midpoint of the pump curve and velocities are in the normal range of 3-5 ft/s for most pipes.

Positive flows show that the assumed flow directions were correct except for pipe 14. Reviews of the nodal heads confirm that flow moves from node 5 to node 9 (Table P5-6e) that is the opposite of the assumed condition resulting in the negative sign on the pipe 14 flow. A negative head loss for the pump indicates a head gain across that link. The total heads for all nodes are presented in Table P5-6e.

7. The new pump was connected to the same end nodes as pump 1. Similarly, the pipe was linked to nodes 1 and 6. As seen in Table P5-7a and b, the parallel pipe and pumps allowed more flow to be supplied from reservoir 10 with more energy. The total head at node 1 increased about 8 ft. Heads throughout the network were raised 6-8 ft.

The flow entering the nodes from the reservoir was 8.02 cfs compared to 4.81 cfs for the single pump. Virtually no flow entered the system from the tank. As a result, the flow pattern changed throughout the network compared to the condition in Problem 5-6. Flow direction changed in several pipes as seen by the negative flow rates. Due to the large pump heads, flow moves toward node 3 (nearest the tank) through pipe 6 supplying that node with the majority of its flow. The tank significantly affects heads in that portion of the network. The operational impact can be determined by closing pipe 15 and examining the change in the system.

8. The PRV's (links 17 and 18) are installed with the downstream node at new nodes (12 and 13) as shown in Figure P5-8.

Table P5-6d: Problem 5-6 pipe results.

Pipe	Flow (cfs)	Velocity (ft/s)	Unit Headloss (ft/kft)
1	1.2	3.44	5.81
2	0.07	0.2	0.03
3	0.75	2.15	2.43
4	2.18	4	5.91
5	0.63	1.16	0.6
6	0.72	1.31	0.75
7	0.25	0.46	0.11
8	1.44	2.65	2.76
9	1.78	2.26	1.67
10	0.68	1.25	0.68
11	0.81	1.03	0.41
12	0.66	1.88	1.9
13	1.32	2.41	2.32
14	0.68	1.25	0.74
15	3.18	4.04	7.43
Pump	4.82	0	-152.12

Table P5-6e: Problem 5-6 node results.

Node	Total Head (ft)
1	152.12
2	146.31
3	146.29
4	143.12
5	142.95
6	145.23
7	145.6
8	146.21
9	142.21
Res. 10	0
Res. 11	150

Table P5-7a: Node results with parallel pumps.

Node	Total Head (ft)
1	160.77
2	152.68
3	150
4	148.06
5	148.05
6	152.46
7	152.63
8	153.68
9	147.84
Res. 10	0
Res. 11	150

Table P5-7b: Problem 5-7 pipe results.

Pipe	Flow (cfs)	Velocity (ft/s)	Unit Headloss (ft/kft)
1	1.43	4.11	8.09
2	0.79	2.26	2.68
3	0.58	1.65	1.49
4	2.4	4.41	7.09
5	0.14	0.26	0.04
6	-1.13	2.07	1.76
7	0.08	0.14	0.01
8	1.6	2.93	3.32
9	1.33	1.7	0.98
10	0.9	1.66	1.16
11	0.55	0.7	0.2
12	0.94	2.69	3.68
13	1.66	3.03	3.55
14	-0.34	0.63	0.21
15	-0.01	0.02	0
18	2.58	3.28	3.32
Pump 1	4.01	0	-160.77
Pump 2	4.01	0	-160.77

For a setting of 50 psi the pressures are reduced at nodes 12 and 13 to the desired 50 psi (Table P5-8a). Energy losses to node 9 reduce the pressure at that node to 49.3 psi. Note the flow rates in pipes 13 and 14 are different due to their different diameters and length but will have the same head loss. Pipe 13 carries 0.93 cfs while pipe 14 has a flow rate of 1.07 cfs.

With the PRV's settings at 65 psi, the valves are fully open since the total heads at nodes 12 and 13 do not have pressures above the permissible 65 psi

(Table P5-8b). The pressure distribution is identical to that computed in Problem 5-6.

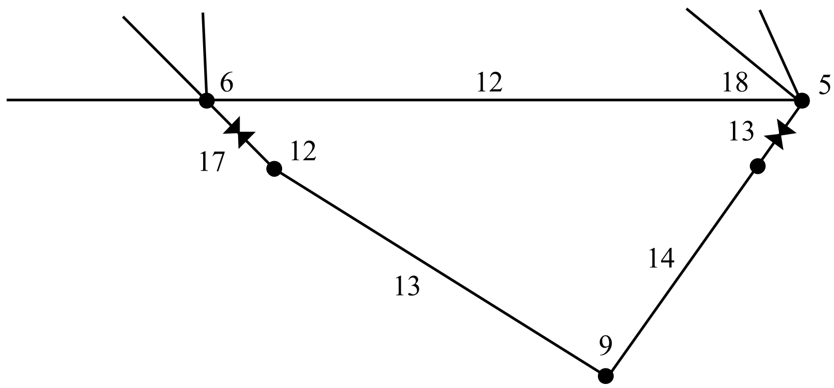


Figure P5-8: Portion of Problem 5-6 network with PRV's to reduce pressure at node 9.

Table P5-8a: Node data for Problem 5-8 with PRV settings at 50 psi.

Node	Demand (cfs)	Total head (ft)	Pressure head (psi)
1	0	152.3	65.99
2	0.5	146.39	63.43
3	0	146.25	63.37
4	0.5	142.54	61.76
5	2	142.27	61.65
6	1	145.54	63.06
7	0.5	145.84	63.19
8	1.5	146.43	63.45
9	2	113.81	49.31
12	0	115.39	50
13	0	115.39	50
Res. 10	-4.81	0	0
Res. 11	-3.19	150	0

9. The maximum fire flow can be determined by trial and error or adding an emitter or reservoir near node 5. A short pipe (10ft with diameter 36 in and C_{HW} = 130) is attached between node 5 and a reservoir with elevation of 69.23 ft (30 psi) as shown in Figure P5-9. This pressure is equal to the required pressure head at node 5.

Under this condition, the reservoir receives a flow of 10.88 cfs and the pressure head at node 5 is 30 psi. This flow rate is the additional flow beyond the 2 cfs that can be supplied under the defined demand condition. All other nodes have pressures exceeding 30 psi.

Table P5-8b: Nodal pressure data for Problem 5-6 network with PRV's set at 65 psi.

Node	Pressure head (ft)
1	65.92
2	63.4
3	63.39
4	62.01
5	61.94
6	62.93
7	63.09
8	63.35
9	61.63
12	62.93
13	61.94

If the fire was located at node 3 and a similar pipe/reservoir was added to that location, a flow of 16.3 cfs could be supplied to node 3 with a pressure of 30 psi. However, several other nodes including nodes 4, 5, and 9 had pressure slightly below 30 psi. Is this condition acceptable? As noted in the earlier discussion, clear definitions of satisfactory, reliable operating conditions must be set when performing this type of analysis.

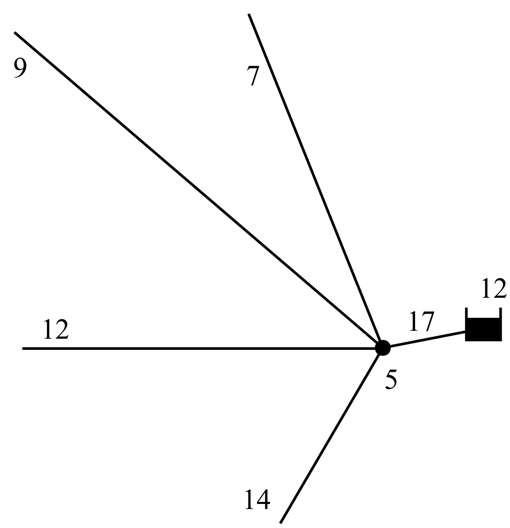


Figure P5-9: Network modification for fire flow.

10. Model results for tank 11 are shown in Table P5-10b. The final tank level is within 0.5 ft of the initial condition. The tank remained open throughout the simulation and the range of tank levels was about 25 ft.

The tank level at time 4:00 can be computed with the flow rate and the initial tank level using Eq. 5-49:

$$H_{T,t+\Delta t} = H_{T,t} + \left(Q_{T,in} - Q_{T,out}\right)\frac{\Delta t}{A_T}$$

where $H_{T,t=0}$ = 150, $Q_{T,in}$ = 1.6 cfs, $Q_{T,out}$ = 0 cfs, Δt = 4 hrs = 14400 s, and $A_T = \pi D_T^2/4 = \pi (50)^2/4 = 1963$ ft^2. Substituting those values above gives:

$$H_{T,t=4hrs} = 150 + (1.6 - 0)\frac{14400}{1963} = 150 + 11.74 = 161.74\,ft$$

The result is slightly off due to rounding of the flow rate in the output table. $Q_{T,in}$ to five decimal places is 1.60054 cfs which provides the exact numerical result shown in Table P5-10b. Other times can be confirmed by a similar analysis.

Table P5-10b: Tank level and flow data for tank 11.

Time (hrs)	Demand (cfs)	Head (ft)
0:00	1.6	150
4:00	-0.84	161.74
8:00	-1.29	155.57
12:00	-1.2	146.14
16:00	1.31	137.37
20:00	0.36	146.94
24:00	1.64	149.58

11. The parallel pumps are links numbered 16 and 17. The following simple rule statements meet the defined conditions. The exact format of these statements varies with the model used.

LINK 16 CLOSED IF NODE 11 ABOVE 155
LINK 17 CLOSED IF NODE 11 ABOVE 145
LINK 17 OPEN IF NODE 11 BELOW 145

The first condition in the problem is that no pumps should operate if the tank level is above 155 ft. Link 16 is off if the level is above 155 ft, and link 17 is off if the level is above 145 ft, so both are off at levels above 155 ft. Since link 16 is on when the level is below 145 ft and pump 17 is off above 145 ft, a single pump operates in the 145-155 ft level range satisfying the second condition. Finally, pump 17 is turned on when the tank level drops below 145ft and both pump operate in this condition.

The resulting tank levels and the pump flow rates are listed in Table P5-11. The tank level begins between 145 and 155 ft and the level begins to fall. Only pump 16 is operating during the first two periods until the level reaches 144.41 ft at time 12:00. Pump 17 is then switched on, but the tank level continues to drop with the higher demands. Both pumps operate until time 20:00 when the level is above 145ft. The tank then fills to a height near the initial level. In this model, the control statements are examined at the hydraulic time step of 4 hours, so the pump switches are not made at the time when the passes 145ft but the next closest time interval.

Table P5-11: Tank levels and pump flows for Problem 5-11 (EPS with controls).

Time (hrs)	Tank 11 level (ft)	Pump 16 flow (cfs)	Pump 17 flow (cfs)
0:00	150.0	4.78	0
4:00	151.84	4.79	0
8:00	150.4	5.14	0
12:00	144.41	4.49	4.49
16:00	136.91	4.67	4.67
20:00	145.75	5.28	0
24:00	149.0	4.85	0

12. Define loops and set $m = 0$. Assume an initial set of pipe flows that satisfy conservation of mass at all nodes. To satisfy conservation of mass with an assumed flow in pipe 10 toward the reservoir requires that the flow is negative (or opposite of the assumed direction). In most cases, one would not begin the solution process with a negative flow but it may occur. More likely, a negative flow will result after a few iterations if the initial assumption was incorrect. As seen below, the mathematics is the same with the addition of verification of the flow direction when computing head losses around a loop.

For the first iteration ($m = 1$), the loop corrections are computed using Eq. 5-20:

$$\Delta Q_{LP} = -\frac{\sum_{l \in lloop} K_l \, Q_l^n}{\sum_{l \in lloop} n \left| h_{L,l} / Q_l \right|}$$

Compute the numerator which is sum of head losses around a loop for each loop substituting $Q^{(0)}$ for Q.

For loop P, the head loss is:

$$h_{L,1} = -K_{10}\lfloor Q_{10}\rfloor^2 + K_9\lfloor Q_9\rfloor^2 - K_5\lfloor Q_5\rfloor^2 - K_2\lfloor Q_2\rfloor^2 + \\ +(240-0.00024\,Q_p^2) - (\Delta E_{8-9})$$

where each term's sign comes from the assumed flow direction and ΔE is the difference in energy between reservoirs 8 and 9. The second sign in the parentheses comes from the direction of the flow rate at the current iteration. The only pipes with negative values in parentheses are those with the incorrect flow directions, i.e., 3, 5 and 10. Substituting the flows in m^3/s:

$$h_{L,1} = -258\,(-1)\,(0.3889)^2 + 2708\,(0.1354)^2 - 2708\,(-1)\,(0.30243)^2 \\ -1088\,(0.0799)^2 + (240-3110.4\,(0.2465)^2) - 200 = \\ +39.05+49.65+1.60-6.94+50.96-200 = -65.68\ m$$

For the closed loops:

$$\sum h_{L,I} = -K_1\lfloor Q_1\rfloor^2 - K_3\lfloor Q_3\rfloor^2 + K_4\lfloor Q_4\rfloor^2 - K_6\lfloor Q_6\rfloor^2 = \\ = -1088\,(0.0556)^2 - 2708\,(-1)(0.0694)^2 + 1088\,(0.0694)^2 - 2708\,(0.0799)^2 \\ = -3.36+13.06+5.25-17.27 = -2.32$$

$$\sum h_{L,II} = K_2\lfloor Q_2\rfloor^2 - K_4\lfloor Q_4\rfloor^2 + K_5\lfloor Q_5\rfloor^2 + K_7\lfloor Q_7\rfloor^2 = \\ = 1088\,(0.0799)^2 - 1088\,(0.0694)^2 \\ +2708\,(-1)(0.0243)^2 + 2708\,(0.059)^2 \\ = 6.94-5.25-1.60+9.43 = 9.52$$

$$\sum h_{L,III} = K_6\lfloor Q_6\rfloor^2 - K_7\lfloor Q_7\rfloor^2 + K_8\lfloor Q_8\rfloor^2 - K_9\lfloor Q_9\rfloor^2 = \\ = 2708\,(0.0799)^2 - 2708\,(0.0590)^2 + 1088\,(0.2014)^2 - 2708\,(0.1354)^2 \\ = 17.27-9.43+44.13-49.65 = 2.32$$

The denominator is the sum of the absolute values of $n\ h_L/Q$. So signs are not of concern in this calculation. Using the values listed in Table P5-12 gives:

$$Loop\ P: n\sum h_{L,P}/Q = n\,K_{10}\,Q_{10} + n\,K_9\,Q_9 + n\,K_5\,Q_5 + n\,K_2\,Q_2 + \\ \left|(2)(-3110)\,Q_p\right| = 200.8+733.3+131.6+173.8+1533.6 = 2773.1$$

$$Loop\ I: n\sum h_{L,P}/Q = nK_1\,Q_1 + nK_3\,Q_3 + nK_4\,Q_4 + K_6\,Q_6 = \\ 120.9+376+151.1+432.5 = 1080.5$$

$$Loop\ II: n\sum h_{L,P}/Q = nK_2\,Q_2 + nK_4\,Q_4 + nK_5\,Q_5 + K_7\,Q_7 = \\ 173.8+151.1+131.6+319.6 = 776.1$$

$$Loop\ III : n\sum h_{L,P} / Q = nK_6\ Q_6 + nK_7\ Q_7 + nK_8\ Q_8 + K_9\ Q_9 =$$
$$432.6 + 319.6 + 438.3 + 733.3 = 1923.8$$

The correction factors for the pseudo-loop is then (in m^3/s):

$$\Delta Q_P = -\frac{\sum_{l \in lloop(P)} K_l\ Q_l^n}{\sum_{l \in lloop(P)} n \left| h_{L,l} / Q_l \right|} = -\frac{-65.68}{2773.1} = +0.024\ m^3/s = 85.3\ m^3/hr$$

The other loops' values are listed in Table P5-12b. Note that the corrections for loops *II* and *III* are negative while loops' *P* and *I* corrections are positive.

Table P5-12b: Summary of calculations for loop corrections for iteration 1.

Loop	*P* (-2-5+9-10+P)	*I* (-1-3+4-6)	*II* (2-4+5+7)	*III* (6-7+8-9)*
$\sum h_{:L}$	-65.68	-2.32	9.52	2.32
$n\sum \left\| h_{:L} / Q \right\|$	2773.1	1080.5	776.1	1923.8
$\Delta Q\ (cms)$	0.024	0.002	-0.012	-0.001
$\Delta Q\ (cmh)$	85.3	7.7	-44.2	-4.3

* Pipes in loop, that is, the set *lloop* for each loop. Signs denote the assumed flow directions.

The pipe flows are then adjusted according to their original assumed flow directions. For example, pipe 1 only appears in loop *I* so it has one correction. Also, since it was assumed to be in the negative direction relative to the loop its correction is negative or:

$$Q_1^1 = Q_1^0 - \Delta Q_I = 200 - (+7.73) = 192.27\ m^3/hr = 0.0534\ m^3/s$$

Similarly for the other pipes,

$$Q_2^1 = Q_2^0 + \Delta Q_{II} - \Delta Q_P = 287.5 + (-44.2) - 85.3 =$$
$$= 158\ m^3/hr = 0.0439\ m^3/s$$
$$Q_3^1 = Q_3^0 - \Delta Q_I = -250 - 7.7 = -257.7\ m^3/hr = -0.0716\ m^3/s$$

$$Q_4^1 = Q_4^0 - \Delta Q_I + \Delta Q_{II} = 200 - (-44.2) + 7.7 =$$
$$= 301.9\ m^3/hr = 0.0839\ m^3/s$$

$$Q_5^1 = Q_5^0 + \Delta Q_{II} - \Delta Q_P = -87.5 + (-44.2) - 85.3 =$$
$$= -217.0\ m^3 / hr = 0.0603\ m^3 / s$$
$$Q_6^1 = Q_6^0 - \Delta Q_I + \Delta Q_{III} = 287.5 - 7.7 + (-4.3) =$$
$$= 275.5\ m^3 / hr = 0.0765\ m^3 / s$$
$$Q_7^1 = Q_7^0 + \Delta Q_{II} - \Delta Q_{III} = 212.5 + (-44.2) - (-4.3) =$$
$$= 172.6\ m^3 / hr = 0.0479\ m^3 / s$$
$$Q_8^1 = Q_8^0 + \Delta Q_{III} = 725 + (-4.3) =$$
$$= 720.7\ m^3 / hr = 0.200\ m^3 / s$$
$$Q_9^1 = Q_9^0 - \Delta Q_{III} + \Delta Q_P = 487.5 - (-4.3) + 85.3 =$$
$$= 577.1\ m^3 / hr = 0.160\ m^3 / s$$
$$Q_{10}^1 = Q_{10}^0 - \Delta Q_P = -1400 - 85.3 =$$
$$= -1485.3\ m^3 / hr = -0.4126\ m^3 / s$$
$$Q_P^1 = Q_P^0 - \Delta Q_P = 887.5 - 85.3 =$$
$$= 802.2\ m^3 / hr = 0.2228\ m^3 / s$$

where the negative signs on pipes 3, 5, and 10 mean that the flow direction is the opposite of the initially assumed direction. The iterations continue and the flows in m^3/h are:

m	**Pipe 1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Pump**
0	200.0	287.5	-250.0	250.0	-87.5	287.5	212.5	725.0	487.5	-1400.0	887.5
1	192.3	158.1	-257.7	301.9	-216.9	275.4	172.6	720.7	577.1	-1485.3	802.2
2	200.7	204.7	-249.3	255.3	-170.3	313.6	181.1	750.4	538.9	-1476.8	810.7
3	183.1	195.7	-266.9	267.7	-179.3	295.5	186.8	749.9	553.7	-1491.0	796.5
4	182.6	204.8	-267.4	258.5	-170.2	305.5	186.0	760.4	543.7	-1491.5	796.0
5	177.0	204.4	-273.0	260.1	-170.6	301.6	188.3	762.0	546.4	-1496.0	791.5
6	175.8	207.0	-274.2	257.6	-168.0	304.0	188.4	765.7	543.9	-1497.0	790.5
7	173.8	207.5	-276.2	257.6	-167.5	303.2	189.2	766.9	544.2	-1498.6	788.9
8	173.1	208.3	-276.9	256.8	-166.7	303.8	189.3	768.3	543.5	-1499.3	788.2
9	172.3	208.6	-277.7	256.7	-166.4	303.7	189.6	768.9	543.5	-1499.9	787.6
10	171.9	209.0	-278.1	256.4	-166.0	303.9	189.7	769.4	543.2	-1500.2	787.3
11	171.6	209.1	-278.4	256.3	-165.9	303.9	189.8	769.8	543.2	-1500.4	787.1
12	171.4	209.3	-278.6	256.2	-165.7	303.9	189.9	770.0	543.1	-1500.6	786.9
13	171.3	209.4	-278.7	256.2	-165.6	303.9	189.9	770.1	543.1	-1500.7	786.8
14	171.2	209.4	-278.8	256.1	-165.6	303.9	189.9	770.2	543.0	-1500.7	786.8
15	171.2	209.4	-278.8	256.1	-165.6	303.9	190.0	770.3	543.0	-1500.8	786.7
16	171.1	209.5	-278.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6
17	171.1	209.5	-278.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6

Thus, seventeen iterations were required to reach convergence. Note that the values are not the same as in Problem 5-3 since the friction factors were assumed to be equal to 0.032 and not determined by the Swamee-Jain equation (or Moody diagram).

13. The simultaneous loop equation method solves the system of equations (Eq. 5-23):

$$\mathbf{J_L}\ \mathbf{\Delta Q} = \mathbf{-F(Q^{(\mathit{m}-1)})}$$

at each iteration *m.*

For the first iteration (m=1) with m^3/s as the flow unit:

$$\mathbf{J_L} = \begin{bmatrix} \partial F_P / \partial(\Delta Q_P) & \partial F_P / \partial(\Delta Q_I) & \partial F_P / \partial(\Delta Q_{II}) & \partial F_P / \partial(\Delta Q_{III}) \\ \partial F_I / \partial(\Delta Q_P) & \partial F_I / \partial(\Delta Q_I) & \partial F_I / \partial(\Delta Q_{II}) & \partial F_I / \partial(\Delta Q_{III}) \\ \partial F_{II} / \partial(\Delta Q_P) & \partial F_{II} / \partial(\Delta Q_I) & \partial F_{II} / \partial(\Delta Q_{II}) & \partial F_{II} / \partial(\Delta Q_{III}) \\ \partial F_{III} / \partial(\Delta Q_P) & \partial F_{III} / \partial(\Delta Q_I) & \partial F_{III} / \partial(\Delta Q_{II}) & \partial F_{III} / \partial(\Delta Q_{III}) \end{bmatrix} =$$

$$= \begin{bmatrix} 2773.1 & 0 & -305.4 & -733.3 \\ 0 & 1080.5 & -151.1 & -432.5 \\ -305.4 & -151.1 & 776.2 & -319.6 \\ -733.3 & -432.5 & -319.6 & 1923.6 \end{bmatrix}$$

where the diagonal terms are the terms in the denominator for each loop in the Hardy Cross method, e.g.:

$$\partial F_P / \partial(\Delta Q_P) = n \sum h_{L,P} / Q$$

$$= n\,K_{10}\,Q_{10} + n\,K_9\,Q_9 + n\,K_5\,Q_5 + n\,K_2\,Q_2 + \left|(2)\,(-3110)\,Q_p\right| =$$

$$= 200.8 + 733.3 + 131.6 + 173.8 + 1533.6 = 2773.1$$

The off-diagonal terms correspond to the absolute value of the sum of the gradients for pipes common to multiple loops, e.g.:

$$\partial F_P / \partial(\Delta Q_{II}) = n \sum \left|h_{L,II} / Q\right|_{(P,II)}$$

$$= -n\left|K_2\,Q_2^{n-1}\right| - n\left|K_5\,Q_5^{n-1}\right| = -173.8 - 131.6 = -305.4$$

This value also appears in column 1 row 3 as the matrix is symmetrical.

The vector, $\mathbf{F}(\mathbf{Q}^{(0)})$, is the numerator terms from the Hardy Cross method or:

$$\mathbf{F}^{\mathbf{T}}(\mathbf{Q}^{(0)}) = [-65.68, -2.32, 9.53, 2.32]^T$$

Solving the system of equations (Eq. 5-23) gives:

$$\Delta\mathbf{Q}^T = [0.0272, 0.0075, 0.0047, 0.0117]^T \ (m^3/s)$$
$$= [98.2, 26.9, 17.0, 42.0]^T \ (m^3/h)$$

The pipe flows are then adjust according to their original assumed flow directions. For example, pipe 1 only appears in loop *I* so it has one correction. Also since it was assumed to be in the negative direction relative to the loop its correction is negative or:

$$Q_1^1 = Q_1^0 - \Delta Q_I = 200 - 26.9 = 173.1\, m^3 / hr = 0.0481\, m^3 / s$$

The other pipes are:

$$Q_2^1 = Q_2^0 + \Delta Q_{II} - \Delta Q_P = 287.5 + 17.0 - 98.2 =$$
$$= 206.3\, m^3 / hr = 0.0573\, m^3 / s$$
$$Q_3^1 = Q_3^0 - \Delta Q_I = -250 - 26.9 = -276.9\, m^3 / hr = -0.0769\, m^3 / s$$
$$Q_4^1 = Q_4^0 - \Delta Q_I + \Delta Q_{II} = 200 - 26.9 + 17.9 =$$
$$= 259.9\, m^3 / hr = 0.0722\, m^3 / s$$
$$Q_5^1 = Q_5^0 + \Delta Q_{II} - \Delta Q_P = -87.5 + 17.0 - 98.2 =$$
$$= -168.7\, m^3 / hr = 0.0469\, m^3 / s$$
$$Q_6^1 = Q_6^0 - \Delta Q_I + \Delta Q_{III} = 287.5 - 26.9 + 42.0 =$$
$$= 302.6\, m^3 / hr = 0.0840\, m^3 / s$$
$$Q_7^1 = Q_7^0 + \Delta Q_{II} - \Delta Q_{III} = 212.5 + 17.0 - 42.0 =$$
$$= 187.5\, m^3 / hr = 0.0521\, m^3 / s$$
$$Q_8^1 = Q_8^0 + \Delta Q_{III} = 725 + 42.0 =$$
$$= 767.0\, m^3 / hr = 0.2131\, m^3 / s$$
$$Q_9^1 = Q_9^0 - \Delta Q_{III} + \Delta Q_P = 487.5 - 42.0 + 98.2 =$$
$$= 543.7\, m^3 / hr = 0.1510\, m^3 / s$$
$$Q_{10}^1 = Q_{10}^0 - \Delta Q_P = -1400 - 98.2 =$$
$$= -1498.2\, m^3 / hr = -0.4162\, m^3 / s$$

$$Q_P^1 = Q_P^0 - \Delta Q_P = 887.5 - 98.2 =$$
$$= 789.32\ m^3\ /\ hr = 0.2193\ m^3\ /\ s$$

These values are significantly different from the first iteration of the Hardy Cross method and the simultaneous loop method converge in only 3 iterations. The values at each iteration are shown below.

Table P5-13: Pipe flow rates in m³/h with iteration number for the simultaneous loop equation solution.

m	Pipe 1	2	3	4	5	6	7	8	9	10	Pump
0	200.0	287.5	-250.0	250.0	-87.5	287.5	212.5	725.0	487.5	-1400.0	887.5
1	173.1	206.3	-276.9	259.9	-168.7	302.6	187.5	767.0	543.7	-1498.2	789.3
2	171.1	209.5	-278.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6
3	171.1	209.5	-278.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6

14. The modified linear method iteratively solves the set of linear equations

$$\mathbf{J_{NL}}\ \mathbf{Q}^{(m)} = \mathbf{F_{NL}} = -\ \mathbf{F} + \mathbf{J_{NL}}\ \mathbf{Q}^{(m\text{-}1)}$$

to update the pipe flow rates.

$\mathbf{J_{NL}}$ is comprised of the derivatives of the node and loop equations with respect to the individual pipe flows. The node equation gradients are 0, +1, and -1. These terms are the coefficients of the flow rates in the conservation of mass equations. Based on the assumed flow directions shown in Figure P5-12, the node equations are written with appropriate signs. For example for node 1, the equation is:

$$Q_1 - Q_3 = 0.125\ \text{m}^3/\text{s}$$

Recall that the resulting flow rate in pipe 3 is 278.9 m³/h (0.0775 m³/s) toward node 1 or –278.9 m³/h. With 0.0475 m³/s toward node 1 in pipe, the negative flow satisfies continuity at the node.

Thus, the coefficients for node 1 are zero except for pipe 1 with a +1 and pipe 3 with a –1. The remainder of the node equations can be written for the network and the coefficients appear in the first *nnode* (7) rows of the Jacobian matrix, $\mathbf{J_{NL}}$, (below).

The last four rows are the gradients of the loop equation with respect to the individual pipe flows with the values for pipe *l* appearing in column *l*. These terms are:

$$\partial F_{LP} \big/ \partial Q_l = n \frac{h_{L,l}}{\lfloor Q_l \rfloor}$$

where the sign is taken from the flow direction relative to the loop, that is, from $\lfloor Q_l \rfloor_{LP}$.

$$
\mathbf{J}_{\mathbf{NL}} = \begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & 0 & -1 & -1 & -1 & 0 \\
0 & -173.8 & 0 & 0 & -131.6 & 0 & 0 & 0 & 733.3 & -200.8 & -1533.6 \\
-120.9 & 0 & -376.0 & 151.1 & 0 & -432.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 173.8 & 0 & -151.1 & 131.6 & 0 & 319.6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 432.5 & -319.6 & 438.3 & -733.3 & 0 & 0
\end{bmatrix}
$$

For pipe 1 in loop I, the value is:

$$
\partial F_I \big/ \partial Q_1 = -n \frac{h_{L,1}}{\lfloor Q_1 \rfloor} = -2 \frac{3.358}{(+0.0556)} = -120.9
$$

where the negative sign is applied based on the assumed flow direction. For pipe 3 in loop I, the value is computed by:

$$
\partial F_I \big/ \partial Q_3 = -n \frac{h_{L,3}}{\lfloor Q_3 \rfloor} = +2 \frac{13.06}{(-0.0694)} = -376.0
$$

where the + sign on the term comes from the assumed clockwise flow and the negative sign on the flow rate denotes that the assumed flow direction is incorrect and is counterclockwise. The remainder of the matrix can be determined as shown above.

The first *nnode* rows of the right hand side vector, $-\mathbf{F} + \mathbf{J}_{\mathbf{NL}}\, \mathbf{Q}^{(m\text{-}1)}$, are the nodal demands. The last *nloop* + *nploop* rows that contain only pipes are:

$$
-F_{LP} + J_{NL,LP}\, Q^{(m-1)} = (n-1) \sum_{l \in lloop} h_{L,l}
$$

or (n-1) times the sum of the head losses around the loop at $Q^{(m\text{-}1)}$, i.e., (n-1) times the residual values. For loop I,

$$
-F_{LP} + J_{NL,LP}\, Q^{(m-1)} = (n-1) \sum_{l \in lloop} h_{L,l} =
$$

$$
(2-1) \sum_{l \in lloop} h_{L,l} = (1)(-2.32) = -2.32
$$

Note that the sign convention for flow relative to the loop is obeyed for each term. With the D-W head loss equation (n = 2), these become the sum of the head losses or the values given in second row of Table P5-12b. The relationship is slightly different for loop P that contains a pump.

$$-F(Q^{m-1}) + \left.\frac{\partial F}{\partial Q}\right|_{Q^{m-1}} Q^{m-1}$$

$$= -\sum (h_L + h_P) - \Delta E \;\; + \sum n\,(h_L / Q^{(0)})\, Q^{(0)} \; + (2\,(-3110.)\,Q_P^{(0)})Q_P^{(0)} =$$

$$-(-65.68) + 2\,(-6.94 + 1.6 + 49.65 + 39.05) + (2(-3110.4)\,(887.5/3600)^2$$

$$= -145.68$$

The resulting RHS vector is then:

$$\mathbf{F_{NL}}(\mathbf{Q}^{(0)}) =$$

$$[0.125, 0.0417, 0.1042, 0.0521, 0.2083, 0.052, 0.052, -145.68, -2.32, 9.52, 2.31]^T$$

The first *nnode* rows of the right hand side vector, - $\mathbf{F} + \mathbf{J_{NL}}\,\mathbf{Q}^{(m\text{-}1)}$, are the nodal demands and the last *nloop* + *nploop* rows are equal to:

$$-F_{LP} + J_{NL,LP}\; Q^{(m-1)} = (n-1) \sum_{l \in lloop} h_{L,l}$$

or (n-1) times the sum of the head losses around the loop at $Q^{(m\text{-}1)}$, i.e., (n-1) times the residual values. With the D-W head loss equation, these become the sum of the head losses or the values given in second row of Table P5-12b.

The resulting RHS is then:

$$-\,\mathbf{F_{NL}}(\mathbf{Q}^{(0)}) =$$

$$[0.125, 0.042, 0.1042, 0.0521, 0.2083, 0.0521, 0.0521, -145.68, -2.32, 9.52, 2.31]^T$$

The resulting set of equations is solved for the unknown pipe flow rates resulting in the values shown in Table P5-14 for iteration 1. Two additional iterations are needed to converge to the solution.

Table P5-14: Flow rates in m^3/h for node-loop equation solution to Problem 5-12.

m	Pipe 1	2	3	4	5	6	7	8	9	10	Pump
1	200.0	287.5	-250.0	250.0	-87.5	287.5	212.5	725.0	487.5	-1400.0	887.5
2	173.1	206.3	-276.9	259.9	-168.7	302.6	187.5	767.0	543.7	-1498.2	789.3
3	171.1	209.5	-278.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6
4	171.1	209.5	-278.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6

15. The node equation formulation iteratively solves:

$$\mathbf{J_N}\,\Delta\mathbf{H} = -\,\mathbf{F_N}$$

where the off-diagonal terms in the Jacobian matrix, $\mathbf{J_N}$, (Eq. 5-30a) are:

$$\left.\frac{\partial F_{N,i}}{\partial(\Delta H_j)}\right|_{H^{m-1}} = -\frac{1}{n\,K_l}\left(\frac{\lfloor H_j - H_i \rfloor}{K_l}\right)^{\left(\frac{1}{n}-1\right)} = -\left|\frac{Q_l}{n\,(H_j - H_i)}\right|$$

and the diagonal terms (Eq. 5-30b) are:

$$\left.\frac{\partial F_{N,i}}{\partial(\Delta H_i)}\right|_{H^{m-1}} = \sum_{l\in ncp(i)} \frac{1}{n\,K_l}\left(\frac{\lfloor H_j - H_i \rfloor}{K_l}\right)^{\left(\frac{1}{n}-1\right)} = \sum_{l\in ncp(i)} \left|\frac{Q_l}{n\,(H_j - H_i)}\right|$$

For node (row) equation 1 (i=1) and connected node (column) 2 (j=2),

$$\left.\frac{\partial F_{N,1}}{\partial(\Delta H_2)}\right|_{H^{(0)}} = -\left|\frac{Q_2}{n(H_2 - H_1)}\right| = -\left|\frac{0.04486}{2(103.66 - 105.85)}\right| = -\,0.01024$$

This terms is also placed in row 2 (node 2) column 1 (connected node 1).

The diagonal term for node 2 is the sum of the gradients for the connecting nodes including the reservoir or:

$$\left.\frac{\partial F_{N,2}}{\partial(\Delta H_2)}\right|_{H^{m-1}} = \sum_{l\in ncp(i)} \left|\frac{Q_l}{n\,(H_j - H_i)}\right| =$$

$$\left|\frac{Q_1}{n\,(H_1 - H_2)}\right| + \left|\frac{Q_2}{n\,(H_3 - H_2)}\right| + \left|\frac{Q_4}{n\,(H_5 - H_4)}\right| + \left|\frac{1}{2\,(3110\,Q_p)}\right| =$$

$$\left|\frac{0.0449}{2(103.66 - 105.85)}\right| + \left|\frac{0.057}{2(102.32 - 105.85)}\right| + \left|\frac{0.0688}{2(100.7 - 105.85)}\right|$$

$$+\left|\frac{1}{2(3110)(0.2153)}\right| = 0.01024 + 0.00807 + 0.00668 + 0.00075 = 0.02574$$

where the last term is the derivative of the pump equation.

The RHS vector $\mathbf{F_N}$ is the residual vector of the conservation of mass equations. For example for node 4:

$$F_4 = Q_8 - Q_6 - Q_3 - q_4 = 0.2125 - 0.0865 - 0.0800 - 0.0521 = -0.0061 \text{ m}^3/\text{s}$$

So $-F_4$ equals 0.0061 in the RHS vector. Here the flow direction in pipe 3 is away from node 4 based upon the nodal head values.

Setting up the remainder of the terms equations results in $\mathbf{J_N}$ and $\mathbf{F_N}$:

$$J_N = \begin{bmatrix} 0.01255 & -0.01024 & 0 & -0.00231 & 0 & 0 & 0 \\ -0.01024 & 0.02574 & -0.00807 & 0 & -0.00668 & 0 & 0 \\ 0 & -0.00807 & 0.01202 & 0 & 0 & -0.00396 & 0 \\ -0.00231 & 0 & 0 & & -0.00213 & 0 & -0.00216 \\ 0 & -0.00668 & 0 & -0.00213 & 0.01232 & -0.0035 & 0 \\ 0 & 0 & -0.00396 & 0 & -0.0035 & 0.00868 & -0.00122 \\ 0 & 0 & 0 & -0.00216 & 0 & -0.00122 & 0.00831 \end{bmatrix}$$

and

$$-\mathbf{F_N}^T = [0.000156, -0.003, 0.000528, 0.00614, 0.000288, 0.000288, 0.02263]^T$$

Solving this system of equations gives the vector of corrections:

$$\Delta\mathbf{H}^T = [4.50, 4.25, 4.40, 5.57, 4.59, 4.57, 4.85]^T$$

and the new heads are computed by $\mathbf{H}^1 = \mathbf{H}^0 - \Delta\mathbf{H}$ to give:

$$\mathbf{H}^{(1)T} = [99.16, 101.60, 97.92, 115.41, 96.10, 103.65, 165.25]^T$$

This process is continued until the heads converge in three iterations as shown in Table P5-15c. Note that the initial solution in this problem was correct in terms of flow directions.

Table P5-15c: Nodal heads for iterations of the nodal head equation solution of Problem 5-12.

m	Node 1	2	3	4	5	6	7
1	103.66	105.85	102.32	120.98	100.70	108.22	170.10
2	99.16	101.60	97.92	115.41	96.11	103.65	165.25
3	99.04	101.49	97.81	115.29	95.99	103.53	165.12
4	99.04	101.49	97.81	115.29	95.99	103.53	165.12

16. The solution of the pipe equations requires iteratively solving:

$$\mathbf{dF_P(Q,H)} = n\,\mathbf{A}_{11}\,\Delta\mathbf{Q}^{(m)} + \mathbf{A}_{12}\,\Delta\mathbf{H}^{(m)} = -\,\mathbf{dE} \qquad \text{(from 5-41)}$$

$$\mathbf{dF_Q(Q,H) = A}_{21}\ \Delta\mathbf{Q}^{(m)} = \mathbf{-\ dq} \qquad \text{(from 5-42)}$$

for $\Delta\mathbf{Q}$ and $\Delta\mathbf{H}$. Then updating $\mathbf{H}^{(m)}$ and $\mathbf{Q}^{(m)}$ by:

$$\mathbf{H}^{(m)} = \mathbf{H}^{(m\text{-}1)} + \Delta\mathbf{H}^{(m)} \qquad \text{(from 5-43)}$$

$$\mathbf{Q}^{(m)} = \mathbf{Q}^{(m\text{-}1)} + \Delta\mathbf{Q}^{(m)} \qquad \text{(from 5-44)}$$

First, we form the connectivity matrix $\mathbf{A}_{12}$,

$$\mathbf{A}_{12} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of $\mathbf{A}_{12}$ correspond to pipes and the columns of the matrix correspond to nodes. The non-zeros are based on the assumed flow directions in Figure P5-12. For pipe 3, the assumed source of flow is node 1 (-1) and the downstream sink is node 4 (1). $\mathbf{A}_{21}$ in Eq. 5-40 is the transpose to $\mathbf{A}_{12}$.

$\mathbf{A}_{11}$ and the RHS are functions of the present values for **H** and **Q**. For m =1, we use the values given in Table P5-12a and P5-15b. $n\mathbf{A}_{11}$ is a diagonal matrix with the diagonal terms being the absolute value of the derivatives of the flow equation. For pipe 1, the equations with appropriate values substituted are:

$$F_{P,1} : -H_2 + H_1 + K_1 \lfloor Q_1 \rfloor^2 = dE_1$$
$$= -105.85 + 103.66 + 1088.1\,(+1)\,(0.0556)^2 = 1.17$$

with gradient of:

$$\frac{\partial F_{P,1}}{\partial Q_1} = n\,K_1 \left|Q_1\right|^{n-1} = 2\,(1088.6)\left|0.0556\right| = 121$$

Since water is assumed to flow from node 2 to 1 and our initial assumption is positive (200 m^3/h = 0.0556 m^3/s), the head balance is as shown with the positive one multiplier defining that direction in the equations. The flow direction in pipe 3, on the other hand, was incorrect which may occur in the initial assumption or during iterations to convergence. For pipe 3, the pipe equation and its derivatives are:

$$F_{P,3} : -H_1 + H_4 + K_3 \lfloor Q_3 \rfloor^2 = dE_3$$
$$= -103.66 + 120.98 + 2707.5\,(-1)\,(0.0694)^2 = 4.28$$

and

$$\frac{\partial F_{P,3}}{\partial Q_3} = n\,K_3\,|Q_3|^{n-1} = 2\,(2707.5)\,|-0.0694| = 376$$

As seen the negative sign is added to the head loss equation for the discharge. Also the derivative term is always positive. The diagonal terms of the $n\,\mathbf{A}_{11}$ for pipes 1-10 and the pump are 121, 174, 376, 151, 132, 432, 320, 438, 722, 200, and 1534.

The residual vector of the head loss equations is:

$$[-\mathbf{dE}]^T = [-1.17 \quad -3.41 \quad -4.26 \quad -0.10 \quad -4.30 \quad 3.01 \quad -1.91 \quad 4.99 \quad 12.2 \quad -0.85 \quad -44.9]$$

and the **dq** vector terms are all equal to zero.

The resulting matrix equations are given below. The set of equations is solved for $\Delta\mathbf{Q}$ and $\Delta\mathbf{H}$ and the result is:

$$\Delta\mathbf{Q}^T = [-0.008 \quad -0.022 \quad -0.008 \quad 0.003 \quad -0.022 \quad 0.004 \quad -0.007 \quad -0.012 \quad 0.016 \quad -0.027 \quad -0.027]$$

and

$$\Delta\mathbf{H}^T = [-3.30 \quad -3.04 \quad -2.53 \quad -4.76 \quad -3.55 \quad -3.86 \quad -4.63]$$

The updated flow rates and nodal heads are computed by Eqs. 5-41 and 5-42 and are listed in Table P5-16a and b for the three iterations required to converge to the solution.

$$
\left[\begin{array}{c|c} n\,\mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{0} \end{array}\right] =
$$

$$
\left[\begin{array}{ccccccccccc|ccccccc}
121 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 174 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 376 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 151 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 132 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 432 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 320 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 438 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 722 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 200 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1534 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Table P5-16a: Flow rates in m^3/h for node-loop equation solution to Problem 5-12.

m	Pipe 1	2	3	4	5	6	7	8	9	10	Pump
1	200.0	287.5	-250.0	250	-87.5	287.5	212.5	725.0	487.5	-1400.	887.5
2	173.1	206.3	-276.9	259.9	-168.7	302.6	187.5	767.0	543.7	-1498.2	789.3
3	171.1	209.5	-276.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6
4	171.1	209.5	-276.9	256.1	-165.5	304.0	190.0	770.4	543.0	-1500.9	786.6

Table P5-16b: Convergence of nodal heads for pipe equation solution.

m	Node 1	2	3	4	5	6	7
1	103.66	105.85	102.32	120.98	100.70	108.22	170.10
2	100.36	102.81	99.79	116.23	97.15	104.36	165.47
3	99.04	101.49	97.81	115.29	95.99	103.53	165.12
4	99.04	101.49	97.81	115.29	95.99	103.53	165.12